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New classes of modified teleparallel gravity models

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ABSTRACT

New classes of modified teleparallel theories of gravity are introduced. The action of this theory is constructed to be a function of the irreducible parts of torsion $f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec})$, where $T_{\rm ax}, T_{\rm ten}$ and $T_{\rm vec}$ are squares of the axial, tensor and vector components of torsion, respectively. This is the most general (well-motivated) second order teleparallel theory of gravity that can be constructed from the torsion tensor. Different particular second order theories can be recovered from this theory such as new general relativity, conformal teleparallel gravity or $f(T)$ gravity. Additionally, the boundary term B which connects the Ricci scalar with the torsion scalar via $R = -T + B$ can also be incorporated into the action. By performing a conformal transformation, it is shown that the two unique theories which have an Einstein frame are either the teleparallel equivalent of general relativity or $f(-T + B) = f(R)$ gravity, as expected.

1. Introduction

General Relativity (GR) is very successful theory accurately describing the dynamics of the solar system. All predictions of general relativity, including gravitational waves, have now been experimentally verified. Nonetheless, when applied to the entire Universe, we are faced with conceptual and observational challenges that are sometimes simply summarised as the dark energy and the dark matter problems. When considering the total matter content of the Universe, it turns out that approximately 95% is made up of these two components we do not fully understand yet. This, together with developments in other fields of physics has motivated a variety of models which can be seen as extensions or modifications of general relativity.

Perhaps surprisingly, alternative formulations of general relativity were constructed and discussed shortly after the formulation of the Einstein field equations. One such description, which is of particular interest to us, is the so-called teleparallel equivalent of general relativity (TEGR). Its equations of motion are identical to those of general relativity, their actions only differ by a total derivative term. While both theories are conceptually different, experimentally these two theories are indistinguishable.

Both theories have been modified in the past which led to the emergence of two popular modified gravity models, namely $f(R)$ and $f(T)$ gravity theories [1–6]. These theories are physically distinct and also have very different characteristic features. The field equations of $f(R)$ gravity are of fourth order while the field equations of $f(T)$ gravity are of second order. The precise relationship between these two distinct theories was recently established in [7] starting from a slightly more general theory which also takes into account a boundary term B. This boundary term is the difference between the Ricci scalar R and the torsion scalar T, $R = -T + B$. It is then possible to build a theory based on $f(T, B)$ that contains both $f(R)$ and $f(T)$ gravities as limits.

Another approach of modifying teleparallel gravity was already considered in 1970s [8] where it was called 'New General Relativity'. In this model the torsion tensor is decomposed into its three irreducible components known as the vector, axial and tensor part. These three pieces are then squared and a linear functional of these squared quantities is considered. For a certain parameter choice, this theory becomes the teleparallel equivalent of general relativity.

In the present paper we are studying a rather natural modification of teleparallel theories of gravity that combines aspects of both $f(T)$ gravity and New General Relativity. We start with squares of the three irreducible components of the torsion tensor which we will denote as $T_{\rm vec}$, $T_{\rm ax}$ and $T_{\rm ten}$, and consider a non-linear functional depending on all three torsion pieces. We can then formulate a novel modified model based on the function

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$f(T_{\rm vec}, T_{\rm ax}, T_{\rm ten})$. It is our main point of this paper to argue that this theory is the most general (well-motivated) modified teleparallel theory and essentially all previously studied teleparallel models can be viewed as its special limiting cases. It is also possible to make connection with theoretical continuum mechanics models which are also formulated using the irreducible torsion pieces, see [9,10].

The main advantage of this general framework is that it allows us to study general properties of teleparallel models. Our analysis of conformal symmetries motivates the introduction of the boundary term and considers a further generalization based on the function $f(T_{\rm vec}, T_{\rm ax}, T_{\rm ten}, B)$. We find then that two unique theories with an Einstein frame are either the teleparallel equivalent of general relativity or $f(R)$ gravity theory.

The notation of this paper is the following: Latin indices denote tangents space coordinate whereas Greek indices denote spacetime coordinates. The tetrad and the inverse of the tetrad are denoted as $e^a{}_\mu$ and $E_a{}^\mu$ respectively.

2. Teleparallel gravity models

Teleparallel theories of gravity are based on the idea of working within a geometrical framework where the notion of parallelism is globally defined. In the standard formulation of general relativity this is only possible for spacetimes which are flat and hence are completely described by the Minkowski metric $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$. When working on manifolds with torsion, it is possible to construct geometries which are globally flat but have a non-trivial geometry.

Let us begin with the tetrad formalism, where the fundamental variable is the tetrad field $e^a{}_\mu$, related to the spacetime metric through the relation

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \,. \tag{1}$$

An additional structure on the manifold is the affine structure, defining the rule of parallel transport, fully characterised by the spin connection. While in General Relativity the connection is assumed to be the torsion-free Levi-Civita connection, in teleparallel gravity the connection is assumed to satisfy the condition of zero curvature

$$R^a{}_{b\mu\nu}(\omega^a{}_{b\mu}) = \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu} \equiv 0 \,, \tag{2}$$

which is solved by the pure gauge-like connection [11,12] given by

$$\omega^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu \Lambda_b{}^c \,, \tag{3}$$

where $\Lambda_b{}^c = (\Lambda^{-1})^c{}_b$. Such spaces are often referred to as Weitzenböck spaces who established the possibility of this construction in the 1920s.

The torsion tensor of this connection

$$T^a{}_{\mu\nu}(e^a{}_\mu, \omega^a{}_{b\mu}) = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu \,, \tag{4}$$

is generally non-vanishing, and transforms covariantly under both diffeomorphisms and local Lorentz transformations. It can be decomposed as follows

$$T_{\lambda\mu\nu} = \frac{2}{3}(t_{\lambda\mu\nu} - t_{\lambda\nu\mu}) + \frac{1}{3}(g_{\lambda\mu} v_\nu - g_{\lambda\nu} v_\mu) + \epsilon_{\lambda\mu\nu\rho} a^\rho \,, \tag{5}$$

where

$$v_\mu = T^\lambda{}_{\lambda\mu} \,, \tag{6}$$

$$a_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho} T^{\nu\sigma\rho} \,, \tag{7}$$

$$t_{\lambda\mu\nu} = \frac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \frac{1}{6}(g_{\nu\lambda} v_\mu + g_{\nu\mu} v_\lambda) - \frac{1}{3} g_{\lambda\mu} v_\nu \,, \tag{8}$$

are three irreducible parts with respect to the local Lorentz group, known as the vector, axial, and purely tensorial, torsions, respectively.

Teleparallel models of gravity are based on the torsion tensor while GR is formulated using the curvature. The most studied teleparallel model is the *teleparallel equivalent of general relativity* (TEGR), or *teleparallel gravity* for short, where the Lagrangian is assumed to take the form

$$\mathcal{L}_{\rm TEGR} = \frac{e}{2\kappa} T \,. \tag{9}$$

The so-called torsion scalar T is defined by[1]

$$T = \frac{3}{2} T_{\rm ax} + \frac{2}{3} T_{\rm ten} - \frac{2}{3} T_{\rm vec} \,, \tag{10}$$

where we have defined three invariants

$$T_{\rm ten} = t_{\lambda\mu\nu} t^{\lambda\mu\nu} = \frac{1}{2}\Big(T_{\lambda\mu\nu} T^{\lambda\mu\nu} + T_{\lambda\mu\nu} T^{\mu\lambda\nu}\Big) - \frac{1}{2} T^\lambda{}_{\lambda\mu} T_\nu{}^{\nu\mu} \,, \tag{11}$$

$$T_{\rm ax} = a_\mu a^\mu = \frac{1}{18}\Big(T_{\lambda\mu\nu} T^{\lambda\mu\nu} - 2 T_{\lambda\mu\nu} T^{\mu\lambda\nu}\Big) \,, \tag{12}$$

$$T_{\rm vec} = v_\mu v^\mu = T^\lambda{}_{\lambda\mu} T_\nu{}^{\nu\mu} \,. \tag{13}$$

The above Lagrangian (9) is equivalent to the Einstein–Hilbert action up to a boundary term. Hence, the TEGR field equations are equivalent to the Einstein's field equations.

The first modified gravity model based on the framework of teleparallel gravity was *new general relativity*, discussed in [8]. It is a natural and simple generalization of Lagrangian (9), where the coefficients in the torsion scalar (10) are assumed to take arbitrary values, this means

$$\mathcal{L}_{\rm NGR} = \frac{e}{2\kappa}\Big(a_0 + a_1 T_{\rm ax} + a_2 T_{\rm ten} + a_3 T_{\rm vec}\Big) \,, \tag{14}$$

where the four a_i are arbitrary constants. The number a_0 can be interpreted as the cosmological constant.

In the recent decade, another straightforward generalization of Lagrangian (9) became increasingly popular after it was shown to be able to explain accelerated expansion of Universe without invoking the dark sector [2–6]. This model is known as *$f(T)$ gravity*, where the Lagrangian is considered to be an arbitrary function of the torsion scalar (10). Its action is given by

$$\mathcal{L}_{f(T)} = \frac{e}{2\kappa} f(T) \,. \tag{15}$$

Another generalization that found applications in cosmology is *teleparallel dark energy*, where the torsion scalar (10) is assumed to be non-minimally coupled to the scalar field [13], or possibly the scalar field is coupled to the vector torsion or the boundary term which relates the Ricci scalar with the torsion scalar [14].

Recently, there has been an increased interest in conformal gravity models that have many attractive features, see for instance [15]. As it turns out, it is possible to construct conformal gravity in the teleparallel framework leading to *conformal teleparallel gravity* [16]. This model has second order field equations, which are much simpler than those of the usual Weyl gravity based on the square of the conformal Weyl tensor. The Lagrangian of this model is taken to be quadratic in the torsion scalar

[1] We remark that it is far more common to write the torsion scalar as $T = \frac{1}{4} T^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu} + \frac{1}{2} T^\rho{}_{\mu\nu} T^{\nu\mu}{}_\rho - T^\lambda{}_{\lambda\mu} T_\nu{}^{\nu\mu}$, which is equivalent to our definition [12]. We follow here the definition in terms of the irreducible parts of the torsion, similarly to NGR [8]. As we will show in Section 4, the main advantage of this approach is the much simpler transformation properties under conformal transformations.