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A scaling approach to evaluating the distance exponent of the urban gravity model

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ABSTRACT

The gravity model is one of important models of social physics and human geography, but several basic theoretical and methodological problems remain to be solved. In particular, it is hard to explain and evaluate the distance exponent using the ideas from Euclidean geometry. This paper is devoted to exploring the distance-decay parameter of the urban gravity model. Based on the concepts from fractal geometry, several fractal parameter relations can be derived from the scaling laws of self-similar hierarchies of cities. Results show that the distance exponent is just a scaling exponent, which equals the average fractal dimension of the size measurements of the cities within a geographical region. The scaling exponent can be evaluated with the product of Zipf's exponent of size distributions and the fractal dimension of spatial distributions of geographical elements such as cities and towns. The new equations are applied to China's cities, and the empirical results accord with the theoretical expectations. The findings lend further support to the suggestion that the geographical gravity model is a fractal model, and its distance exponent is associated with a fractal dimension and Zipf's exponent. This work will help geographers understand the gravity model using fractal theory and estimate the distance exponent using fractal modeling.

1. Introduction

A set of gravity models have been applied to explain and predict various behaviors of spatial interactions in many social sciences. Among this family of models, the basic one is the gravity model of migration based on an inverse power-law distance-decay effect, which was proposed by analogy with Newton's law of gravitation. It can be used to describe the strength of interaction between two places [35,43,51,54]. According to the model, any two places attract each another by a force that is directly proportional to the product of their sizes and inversely proportional to the bth power of the distance between them, and b is the distance-decay exponent, which is often termed *distance exponent* for short [32]. The size of a place can be measured with appropriate variables such as population numbers, built-up area, and gross domestic product (GDP). Recently, the gravity model has been employed to study the attractive effect of various new-fashioned human and physical activities by means of modern technology [2,31,37–40,42,44,56]. The model is empirically effective for describing spatial interactions, but it is obstructed by two problems on the distance exponent, b. One is the dimensional problem, that is, it is impossible to interpret the distance exponent in light of Euclidean geometry [32,34]. The other is the algorithmic problem, namely, it is hard to estimate the numerical value of the distance exponent [48].

The first problem can be readily solved by using ideas from fractals. Fractal geometry was developed by Mandelbrot [47], and it is a powerful tool for spatial analysis in geographical research [4,7,27,28]. It can be proved that the distance exponent is a fractal parameter indicating a space dimension [16]. However, how to evaluate the distance exponent is still a difficult problem that remains to be solved. A new discovery is that the distance exponent is associated with the Zipf exponent of the rank-size distribution and the fractal dimension of a self-organized network of urban places [11]. If we examine the spatial interaction between cities and towns in a region, the distance exponent of the gravity model is just the product of the Zipf exponent of the city-size distribution and the fractal dimension of the corresponding central-place network. The formula has been derived by combining central-place theory and the rank-size rule [11]. A pending problem is how to use the formula to estimate the distance exponent for gravity analysis in empirical studies.

This problem can be solved using a self-similar hierarchy with cascade structure. On the one hand, the rank-size distribution is mathematically equivalent to the hierarchical structure [12,13]. On the other, the hierarchical structure and the network structure represent two different sides of the same coin [7]. In short, the rank-

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size distribution and the network structure can be linked with one another by hierarchical structure. Thus, the Zipf exponent of the rank-size distribution and the fractal dimension of spatial network can be associated with each other through hierarchical scaling, which suggests a new approach to estimating the distance exponent. This study is mainly based on urban geography and is devoted to exploring the methods for distance exponent estimation. The rest of this article is organized as follows. In Section 2, a scaling relation will be derived from the gravity model based on the inverse power law, and spatial scaling will be transformed into hierarchical scaling in terms of the inherent association between hierarchy and network. Three scaling approaches to estimating the distance exponent will be proposed. In Section 3, Chinese cities will be employed to make an empirical analysis by means of census data and spatial distance data. In Section 4, several related questions will be discussed. Finally, the discussion will be concluded by summarizing the main points in the work.

2. Theoretical results

2.1. Breaking-point relation

The new method can be derived from the well-known breaking-point formula based on the basic gravity model. Consider three locations within a geographical region, *i*, *j*, and *x*, and *x* falls between *i* and *j*. If there are cities or towns at these locations, the "mass" of the settlements can be measured by population or other size quantity. Thus the gravity can be expressed as

$$I_{ix} = G\frac{P_i P_x}{L_{ix}^b}, \tag{1}$$

$$I_{jx} = G\frac{P_j P_x}{L_{jx}^b}, \tag{2}$$

where I_{ix} denotes the gravity between locations *i* and *x*, I_{jx} denotes the gravity between location *j* and *x*, L_{ix} indicates the distance between locations *i* and *x*, L_{jx} indicates the distance between locations *j* and *x*, P_i, P_j, and P_x are the sizes of the cities at locations *i*, *j*, and *x*, *G* refers to the gravity coefficient, and *b* to the distance exponent indicative of spatial friction. Combining Eqs. (1) and (2) yields

$$\frac{P_i/L_{ix}^b}{P_j/L_{jx}^b} = \frac{I_{ix}}{I_{jx}}. \tag{3}$$

Suppose that there exists a special location for *x*, where $I_{ix} = I_{jx}$. In this case, eliminating *x* yields

$$\frac{P_i}{P_j} = \left(\frac{L_{ix}}{L_{jx}}\right)^b, \tag{4}$$

which is familiar to geographers. Chen [16] demonstrated that Eq. (4) represents a fractal dimension relationship. Eq. (4) is identical in form to the well-known breaking-point formula derived by Reilly [50] and Converse [23]. By using the hierarchical scaling laws of urban systems, we can reveal the spatial meaning of the parameter *b* and find a new way of evaluating it.

2.2. Distance exponent based on hierarchical scaling laws

If a geographical region is large enough, the size distribution of cities within the region may be consistent with Zipf's law. However, the laws of complex social and economic systems are not of spatio-temporal translational symmetry, and city-size distributions do not follow the only law [17]. Sometimes, Zipf's law is replaced by other mathematical laws such as Lavalette law [9]. Zipf's law is one of the well-known rank-size scaling law, indicating self-organized criticality of urban evolution [1,19]. Suppose that city development in a region fall into the self-organized critical state. The general Zipf formula can be expressed as follows [67]

$$P(k) = P_1 k^{-q}, \tag{5}$$

where *k* refers to the rank of cities in descending order, $P(k)$ to the size of the city of rank *k*, *q* denotes the Zipf scaling exponent, and the proportionality coefficient P_1 indicates the size of the largest city in theory. City size is always measured with resident population. The inverse function of Eq. (5) is $k = [P(k)/P_1]^{-1/q}$, in which the rank *k* represents the number of cities with size greater than or equal to $P(k)$. Reducing $P(k)$ to *P* and substituting *k* with $N(P)$, we have

$$N(P) = \eta P^{-p}, \tag{6}$$

which can be converted into the function of Pareto's density distribution. This implies that Zipf's law is equivalent in mathematics to Pareto's law and that the Zipf distribution is theoretically equivalent to the Pareto distribution [14]. In Eq. (6), the coefficient $\eta = P_1{}^p$ refers to the proportionality constant, and the power exponent $p = 1/q$ denotes the Pareto scaling exponent, indicating the fractal dimension of city rank-size distributions. The reciprocal relation between *p* and *q* is based on pure mathematical derivation. In empirical analyses, this relation proved to be replaced by $p = R^2/q$, where *R* denotes Pearson correlation coefficient [17,18,59].

The rank-size distribution is actually a signature of the self-organized hierarchy with cascade structure. It has been demonstrated that if the city-size distribution follows Zipf's law, the cities can be organized into a self-similar hierarchy [12,13]. The hierarchy of cities consisting of *M* classes (levels) in a top-down order can be described with the discrete expressions of two exponential functions as below:

$$N_m = N_1 r_{\mathrm{n}}^{m-1}, \tag{7}$$

$$P_m = P_1 r_{\mathrm{p}}^{1-m}, \tag{8}$$

where $m = 1, 2, \ldots, M$ refers to the order of classes in the hierarchy of cities (*M* is a positive integer), N_m and P_m denote the number of cities and average size of the cities in the *m*th class, N_1 and P_1 are the number and mean size of the cities in the top class, and $r_{\mathrm{n}} = N_{m+1}/N_m$ and $r_{\mathrm{p}} = P_m/P_{m+1}$ are the *number ratio* and *size ratio* of cities, respectively. If $r_{\mathrm{n}} = 2$ as given, then the value of r_{p} can be calculated; if $r_{\mathrm{p}} = 2$ as given, the value of r_{n} can be derived [13,24]. According to the central-place theory propounded by Christaller [20], we have the third exponential model such as [11]

$$L_m = L_1 r_{\mathrm{l}}^{1-m}, \tag{9}$$

where L_m denotes the average distance between two urban places in the *m*th class, L_1 is the average distance between the urban places in the top class, and $r_{\mathrm{l}} = L_m/L_{m+1}$ is the *distance ratio* of cities (the subscript of *L* is the number 1, and the subscript of *r* is the letter *l*). Here we assume there more than two cities at the first level of an urban hierarchy. If there are just two cities in the first class, L_1 will refer to the distance between the two cities; if there is only one top-level city, L_1 can be substituted by the radius of the equivalent circle of a study area.

The fractal model and hierarchical scaling relations can be derived from the three exponential functions. From Eqs. (7) and (8), the size-number hierarchical scaling relation can be derived as

$$N_m = \mu P_m^{-p}, \tag{10}$$

which is equivalent to Pareto's law, Eq. (6) [see Appendix 1]. The proportionality coefficient is $\mu = N_1 P_1{}^p$, and the scaling exponent