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Kinetic hierarchy and propagation of chaos in biological swarm models

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ABSTRACT

We consider two models of biological swarm behavior. In these models, pairs of particles interact to adjust their velocities one to each other. In the first process, called 'BDG', they join their average velocity up to some noise. In the second process, called 'CL', one of the two particles tries to join the other one's velocity. This paper establishes the master equations and BBGKY hierarchies of these two processes. It investigates the infinite particle limit of the hierarchies at large time scales. It shows that the resulting kinetic hierarchy for the CL process does not satisfy propagation of chaos. Numerical simulations indicate that the BDG process has similar behavior to the CL process.

1. Introduction

The derivation of kinetic equations from particle models of swarming behavior has recently received a great deal of attention. In biological swarm modeling, the most widely used models are particle ones (also known as 'Individual-Based Models') [1–5]. However, to investigate the large scale behavior of biological systems such as fish schools or insect swarms, kinetic [6–10] and hydrodynamic [2,11,12] models have proved to be valuable alternatives. The question of showing a rigorous link between the particle and kinetic levels is mostly open. In [13], a mean-field limit of the Vicsek particle model [5] is performed and leads to a nonlinear Fokker–Planck equation proposed in [14]. A similar program has been performed for the Cucker–Smale model [15,7].

In this work, we investigate two examples of particle systems which are representative of swarming behavior: the so-called BDG and CL processes. These two processes mimic consensus formation in biological groups in a similar way as the Vicsek alignment interaction does [5]. But they describe consensus formation by means of binary interactions, instead of mean-field type interactions like in [5]. In the first process, called 'BDG' (after Bertin et al. [16,17]), two interacting particles join their average velocity up to some noise. In the second process, called 'CL' (for 'Choose the Leader' [18]), one of the two particles tries to join the other one's velocity up to some noise. In [16,17], Bertin et al. formally derive a kinetic description of the BDG dynamics where the interactions are described through a Boltzmann-like collision operator. In [18], the rigorous derivation of kinetic equations for a large class of binary interaction processes including the BDG and CL dynamics is performed in a space-homogeneous setting. The derivation is based on the proof that these systems satisfy the so-called 'propagation of chaos property'. However, this property is proven on a time scale such that each particle collides only a finite number of times. Such a time scale is called the 'kinetic time scale'.

The goal of the present paper is to investigate whether the propagation of chaos property holds for the BDG and CL processes on larger time scales. Large time scales are those which are relevant for hydrodynamic models. Indeed, in order to pass from kinetic to hydrodynamic equations, a hyperbolic rescaling of the kinetic equation is needed. This rescaling consists in changing the time scale to larger ones [19] (it also involves a change of the spatial scale but in the present work, we ignore the spatial variable). If the kinetic equation is proved valid at the kinetic scale but invalid at larger time scales, it is not clear that such a rescaling is meaningful. That propagation of chaos holds at large time scales compared to the kinetic one is central for the validity of hydrodynamic models. This question was already touched upon for the CL dynamics in [18], where the invariant measure (i.e. the stationary state) is shown to violate the chaos property. In the present work, we further elaborate on this question. In particular, we prove that the violation of the chaos property for the CL process happens in the course of the dynamics and not only for the stationary state. We also provide clues that the BDG process must behave in a similar way.

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However, to investigate scales larger than the kinetic ones at the particle level, the dynamics must also be rescaled in some way. In the CL dynamics, if no rescaling of the noise is performed, the large-scale dynamics is then dominated by the noise. The noise will ultimately destroy any directional coherence and will not allow for a hydrodynamic regime to build up. The appropriate rescaling consists in letting the variance of the noise tend to zero like the reciprocal of the particle number. In the BDG dynamics, where consensus making (expressed by the choice of the common average direction) and the noise play more symmetrical roles, a mere rescaling of the noise is not sufficient. For this reason, we consider a 'biased BDG' dynamics, where the collision probability depends on the relative velocities of the particles. Then, the rescaling also involves a grazing collision limit, i.e. having the collision occur only if the relative velocities of the two particles are small. In both cases, the small noise regime precisely corresponds to the emergence of collective motion. The hydrodynamic description, which supposes a non-zero bulk fluid velocity is only relevant if collective motion develops. Therefore, the large time scale validity of propagation of chaos is crucial for the establishment of hydrodynamic models of collective motion. This is the question which is addressed here.

In this paper, we focus on space-homogeneous problems and ignore the spatial variables. Consequently, interactions may happen among any pair of individuals in the pool with a certain probability. We also assume that the individuals move in a two-dimensional space with unit speed. The state of each particle is described by its velocity vector v on the one-dimensional sphere $\mathbb{S}^1$. The state of an N-particle system can be described by its N-particle probability F_N. In the present framework, F_N is a function of the N velocity coordinates $(v_1, \dots, v_N)$ on the torus $\mathbb{T}^N = (\mathbb{S}^1)^N$ and of time. The particle dynamics translates into a time-evolution equation for F_N called the 'master equation'. In [18], we have investigated the class of 'pair-interaction driven' master equations, of which the BDG and CL master equations are members. We have shown that, as $N \to \infty$, propagation of chaos holds. A propagation of chaos result states that the solution $F_N(t)$ can be approximated (in a sense to be defined below) by an N-fold tensor product of the single-particle distribution $F_1(t)$ provided that this property is true initially. This means that the particles become nearly independent and that the system can be described by its single-particle distribution $F_1(t)$ instead of the N-particle distribution F_N. The dimension of the problem is therefore considerably reduced.

To investigate the large N limit, it is difficult to work with F_N alone. Indeed, the limit of F_N as $N \to \infty$ is literally a function of an infinite number of variables. The functional treatment is simplified by considering the k-particle marginal $F_{N,k}$, which is the joint probability of any subset of k particles. The number of variables involved in $F_{N,k}$ is k and stays fixed as $N \to \infty$. The drawback of this method is that the equation satisfied by $F_{N,k}$ depends on the other marginals $F_{N,k'}$ in general. Thus, the equations for the $(F_{N,k})_{k\in\{1,\dots,N\}}$ are all coupled together, forming the so-called BBGKY hierarchy [19]. When $N = \infty$, the hierarchy involves an infinite number of coupled equations and is called the kinetic hierarchy. Showing a propagation result in the limit $N \to \infty$ involves breaking the coupling between the equations in the kinetic hierarchy in some way.

Consensus formation in swarm models should be associated with the build-up of correlations between the particles over time. The fact that the BDG and CL models, as a result of [18], satisfy a propagation of chaos result is counter-intuitive. The resolution of this paradox lies in the investigation of time scales. Indeed, the result of [18] is only valid on finite time intervals at the kinetic scale. On this time scale, the number of collisions undergone by each particle is bounded independently of N. The present paper investigates whether correlation build-up happens at larger time scales. As described above, the investigation of such large time scales is necessary in view of the establishment of hydrodynamic models. The investigation of these large time scales at the level of the particle system requires some rescaling of the processes as described above.

The main objective of this paper is to establish the kinetic hierarchies for the rescaled BDG and CL processes. We then investigate whether these hierarchies possess solutions which satisfy propagation of chaos. For the CL hierarchy, we show that it is never the case. In [18], it was already established that the invariant densities (i.e. the stationary solutions) do not satisfy propagation of chaos. This was done by looking at the single and two-particle marginals only. Here, we extend [18] by showing that the time-dependent solution of the CL hierarchy never satisfies propagation of chaos either. We also provide a general formula for the k-particle marginal invariant density.

Concerning the BDG dynamics, the situation is unclear, in spite of the apparent simplicity of the hierarchy equations. We notice that uniform densities are stationary solutions of the BDG hierarchy. However, the question of uniqueness of stationary solutions for this hierarchy is open. There might exist other solutions which do not satisfy the chaos property. From [18], we know that propagation of chaos is true and that the kinetic equation is valid on the kinetic time scale. The uniform distribution is clearly a stationary solution of this kinetic equation. However, in [17], it is shown that this equilibrium is linearly unstable if the noise level is small enough (see also [20]). This suggests the existence of a second class of anisotropic equilibria (similar to the Von-Mises equilibria of [21,22]). In relation to this other class of equilibria, there may exist solutions of the rescaled kinetic hierarchy other than the uniform distributions. These other solutions may not satisfy the chaos property.

To improve our understanding, we use numerical experiments. We generate the stationary one and two particle marginals by running a large number of independent time-dependent simulations of the particle dynamics. The experimental results concerning the CL dynamics consolidate the theoretical findings. In particular, the theoretical and numerical two-particle correlations show remarkably good agreement. The experimental study of the BDG dynamics shows a similar behavior to the CL dynamics. For this reason, it should be expected that the BDG dynamics lacks chaos property on the large time scale. However, a rigorous result in this direction is not available yet.

The breakdown of propagation of chaos at time scales larger than the kinetic one (which is a proven fact in the CL case and a conjecture in the BDG case) has important consequences. Indeed, such breakdown indicates that kinetic models may not be valid at the hydrodynamic scale and may not be usable to derive hydrodynamic models. In such a case, alternate types of models or at least, deep modifications of classical kinetic models may be needed. Finding such models is a fully open problem. This question is even more complex in the space-inhomogeneous case (e.g. for the modeling of a particle swarm), where an analogous result to [18] is still lacking. The hydrodynamic model of [16,17] is developed under the assumption of a small perturbation to an isotropic equilibrium. As long as this assumption remains valid, the underlying kinetic hierarchy is close to being factorized. In this regime, propagation of chaos is valid, at least approximately, and so is the hydrodynamic model. However, if the perturbation to the isotropic equilibrium becomes larger, propagation of chaos breaks down and the validity of the model in this range may be questioned.

In the literature, propagation of chaos has been mainly investigated in the context of the Boltzmann equation and its caricature proposed by Kac. Early works involve the names of Kac, Lanford, McKean and others [23–25]. They have initiated a considerable activity [26–28]. A new approach has been recently

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