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Exact transparent boundary conditions for the parabolic wave equations with linear and quadratic potentials

R.M. Feshchenko [a,*], A.V. Popov [b]

[a] P.N. Lebedev Physical Institute of RAS, 53 Leninski Pr., Moscow, 119991, Russia

[b] Pushkov Institute of Terrestrial Magnetism, Ionosphere and Radiowave Propagation of RAS, Troitsk, Moscow region, 108840, Russia

HIGHLIGHTS

- Exact transparent boundary conditions for the parabolic wave equation are proposed.
- The dialectic permittivity depends linearly or quadratically on the coordinate.
- The integral kernels of these conditions contain only elementary functions.
- The conditions simplify numerical solution of the parabolic equation in open domains.

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ABSTRACT

In this paper exact 1D transparent boundary conditions (TBC) for the 2D parabolic wave equation with a linear or a quadratic dependence of the dielectric permittivity on the transversal coordinate are reported. Unlike the previously derived TBCs they contain only elementary functions. The obtained boundary conditions can be used to numerically solve the 2D parabolic equation describing the propagation of light in weakly bent optical waveguides and fibers including waveguides with variable curvature. They also are useful when solving the equivalent 1D Schrödinger equation with a potential depending linearly or quadratically on the coordinate. The prospects and problems of discretization of the derived transparent boundary conditions are discussed.

1. Introduction

Leontovich and Fock introduced the parabolic wave equation (PWE) (also known as the Fresnel equation) about fifty years ago [1]. The PWE is widely used in radiophysics and oceanic acoustics for modeling electromagnetic wave and sound propagation [2–4]. An important application of PWE has been found in visible light and X-ray optics, where PWE is used to describe the propagation of weakly divergent light beams in inhomogeneous media [5]. The PWE can be also used to describe coherent scattering phenomena in X-ray imaging [6].

A lot of papers have been dedicated to the refinement of the parabolic equation method in order to increase its numerical accuracy and adjust it to specific physical problems (wide-angle version [7], vectorial PWE [8], etc.). However, there are important applications where the original Leontovich's approximation still provides excellent accuracy and numerical efficiency. In particular, in hard X-ray optics, where optical constants are close to unity, Leontovichs PWE is a powerful tool for modeling complex optical structures such as nano-waveguides [9] and diffraction zone plates [5,10].

* Corresponding author.
E-mail addresses: rusl@sci.lebedev.ru (R.M. Feshchenko), popov@izmiran.ru (A.V. Popov).

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No matter what kind of parabolic equation is concerned, any numerical solution, for instance, by a finite difference (FD) approximation, requires appropriate boundary conditions (BC) as realistic computational domains are necessary finite, while the original wave field might be sought in the whole space. Such a BC must substitute accurate wave field calculations outside the finite computational domain with some relations between its boundary values [11].

There are two main approaches to the computational domain truncation for the parabolic type equations. The first one is based on an exact analytical solution of the governing PE in the outer domain, free of diffracting objects. Projection of such a solution onto the computational domain border leads to so called transparent boundary conditions (TBC) [12,13]. Completely different idea underlies the perfectly matched layer (PML) techniques [14,15]: the assumption that a gradual change of the medium parameters by adding small absorption will cause the outgoing wave attenuation without producing backward reflection. Both approaches have their advantages and shortcomings: the former one is in a sense exact but may involve sophisticated derivations whereas the letter is analytically simpler but basically of approximate nature. In this paper we will only deal with the former type, which is based on assumption that any wave that reaches the boundary from within of the computational domain propagates outward and never returns. TBCs are generally non-local Neumann-to-Dirichlet or Dirichlet-to-Neumann mappings relating the wave field amplitude boundary values with its first derivative a coordinate.

For a general review of TBCs and their applications see [11]. Here we confine ourselves with the original Leontovich 2D PWE. For this equation, i.e. Eq. (1) (see below) with $\alpha = 0$, a TBC was formulated about twenty years ago and now is known as Baskakov–Popov–Papadakis (BPP) condition [12,4,13,16]. A TBC for the linear potential (see Eq. (2) below) but for constant coefficient $g(z) = 1$ has also been known for a long time [13], although, involving integration of a ratio of Airy functions to obtain the kernel, it is quite complicated and poses difficulties for the numerical implementation [17]. In this paper, extending our previous work [18] we contrive to obtain two TBCs for the 2D PWE: (i) one for the linear potential with varying coefficient $g(z)$ (ii) and one for the quadratic potential (see Eq. (3) below). These TBCs do not involve special functions and additional integration, having kernels explicitly expressed via elementary functions. We also try to preserve as much generality as possible by considering in the linear potential case the curvature $g(z)$ to be an arbitrary positive function of z.

2. Methods

In this paper we will only be concerned with the linear 2D PWE having the following form

$$2ik\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} + k^2\alpha(x,z)u = 0, \tag{1}$$

where x and z are coordinates, $k = 2\pi/\lambda$ is the wave number and $\alpha(x,z)$ is a finite function. The computational domain is defined as $-x_0 < x < x_0$ where x_0 is a positive number. We will also assume that $u(0,x) = 0$ when $|x| > x_0$. The Eq. (1) is a full analog of the 1D Schrödinger equation (SE), where z is replaced with time and α is the potential.

The function $\alpha(x,z)$ generally comprises both on the dielectric permittivity of the medium and on a particular choice of the coordinate system [19]. For instance, in a weakly bent optical waveguide or fiber we can introduce a curvilinear system of coordinates so that the propagation of light is described by the ordinary PWE but with a fictitious dielectric permittivity, which is a sum of the true dielectric permittivity $\alpha_0(x,z)$ and an additional term resulting from this particular choice of coordinates.

We will consider two cases for the dependence of $\alpha(x,z)$ on the transversal coordinate x. The first case is that of the linear dependence with the coefficient itself dependent on z as is shown in the following expression

$$\alpha(x,z) = \alpha_0(x,z) + ag(z)x, \tag{2}$$

where $\alpha_0(x,z)$ is a function having a compact support, i.e. $\alpha_0(x,z) = 0$ outside the computational domain when $|x| > x_0$, a is a constant and $g(z)$ is a real positive function. So, outside the computational domain we have the PWE with linear dependence of $\alpha(x,z)$ on coordinate x. It can be shown that in the case of weak bending $g(z) = 2/R(z)$ in the second term of Eq. (2). In other words $g(z)$ is the curvature of the waveguide or fiber and $R(z)$ is the curvature radius.

In the second case function $\alpha(x,z)$ can be expanded further up to the second order by coordinate x as

$$\alpha(x,z) = \alpha_0(x,z) + bx^2, \tag{3}$$

where $\alpha_0(x,z)$ has a compact support as in the previous case and b is a parameter. In (3) we omitted the linear term as it can be always eliminated by a simple shift of the coordinate system. The 2D PWE with the quadratic dependence of the potential term on a coordinate may have applications in the waveguide theory in optics and acoustics. In case of the quantum mechanics and 1D SE the quadratic potential describes a particle in a parabolic trap (in case the potential is attractive), which is itself an interesting problem, and which numerical solution can be greatly facilitated by an appropriate TBC.

We will derive exact TBC from Eq. (1) with both linear and quadratic dependence of α on x by applying to it the Laplace transform by coordinate x. The transformed equations will be linear and of the first order relative the partial derivatives and susceptible to the solution by standard methods of the mathematical physics [20]. The final result is obtained by applying the reverse Laplace transform to the obtained solutions.