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ORIGINAL ARTICLE

Lévy stable distribution and space-fractional Fokker–Planck type equation

Jun-Sheng Duan [a,*], Temuer Chaolu [b], Zhong Wang [c], Shou-Zhong Fu [c]

[a] *School of Sciences, Shanghai Institute of Technology, Shanghai 201418, PR China*
[b] *College of Sciences and Arts, Shanghai Maritime University, Shanghai 200135, PR China*
[c] *Department of Mathematics, Zhaoqing University, Zhaoqing, Guang Dong 526061, PR China*

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Abstract The space-fractional Fokker–Planck type equation $\frac{\partial p}{\partial t} + \gamma \frac{\partial p}{\partial x} = -D(-\Delta)^{\alpha/2} p$ $(0 < \alpha \leqslant 2)$ subject to the initial condition $p(x, 0) = \delta(x)$ is solved in terms of Fox H functions. The solution as $\gamma = 0$ expresses the Lévy stable distribution with the index α. From the properties of Fox H functions, the series representation and asymptotic behavior for the solution are also obtained. Lévy stable distribution as $0 < \alpha < 2$ describes anomalous superdiffusion and its diffusion velocity is characterized by $x_d \propto (Dt)^{1/\alpha}$.

1. Introduction

In recent decades, the theory and applications of the fractional calculus have developed rapidly. The applied fields include viscoelasticity, anomalous diffusion, heat transfer, signal processing, dynamics and control, and so on (Podlubny, 1999; Kilbas et al., 2006). Different definitions and methods have also been proposed (Yang et al., 2013; Yang and Baleanu, 2013; Yang, 2012; Li et al., 2011a,b). Scientists and engineers have found the description of some phenomena is more accurate when the fractional derivative is used. In particular, anomalous diffusion can be characterized by fractional differential equations.

Let random variable $X(t)$ denote the location of diffusing particle with $X(0) = 0$ and $p(x, t)$ be the probability density function for $X(t)$. The time-fractional Fokker–Planck type equations are derived and solved in Hilfer (1995) and Rangarajan and Ding (2000); Space-fractional Fokker–Planck type equation is obtained in Compte (1996) and Yanovsky et al. (2000) using statistical methods. One-dimensional case with convective term without asymmetric term reads (Yanovsky et al., 2000)

$$\frac{\partial p}{\partial t} + \gamma \frac{\partial p}{\partial x} = -D(-\Delta)^{\alpha/2} p, \tag{1}$$

with the initial condition

$$p(x, 0) = \delta(x). \tag{2}$$

Here α, γ, D are real constants $(0 < \alpha \le 2, D > 0)$, $\delta(x)$ is the Dirac delta function and fractional Laplace operator is defined by

* Corresponding author.
E-mail address: duanjs@sit.edu.cn (J.-S. Duan).
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$$(-\Delta)^{\alpha/2}f(x) = \mathcal{F}^{-1}[|k|^{\alpha}\mathcal{F}[f(x)]], \tag{3}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote, respectively, the Fourier transform and its inverse:

$$\mathcal{F}[f] = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}dx, \tag{4}$$

$$\mathcal{F}^{-1}[\hat{f}] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{-ikx}dk. \tag{5}$$

Space-fractional Fokker–Planck type equations similar to the Eq. (1) are also considered in Chaves (1998) and Chechkin et al. (2002), but therein the exact analytic solutions for the problem (1) and (2) are given only for the cases of $\alpha = 1$ and $\alpha = 2$. In the general case of $0 < \alpha \le 2$, the solution for the problem (1) and (2) is studied in the sequel. We obtain the exact analytic solution in terms of Fox H functions (Mathai and Saxena, 1978; Srivastava et al., 1982), and its series representation and asymptotic behavior are investigated.

2. Solution to the problem

Taking the Fourier transform for the problem (1) and (2) with respect to x we get

$$\hat{p}(k,t) = e^{i\gamma tk}e^{-Dt|k|^{\alpha}}. \tag{6}$$

Let

$$U(x,t) = \mathcal{F}^{-1}[e^{-Dt|k|^{\alpha}}]. \tag{7}$$

Then the inverse Fourier transform of (6) is

$$p(x,t) = U(x-\gamma t, t). \tag{8}$$

Lévy stable distribution $\rho(x)$ with index γ $(0 < \gamma \le 2)$ is defined through the Fourier transform as (Feller, 1971; Fogedby et al., 1992; Zanette, 1997)

$$\mathcal{F}[\rho(x)] = \hat{\rho}(k) = \exp(-|k/k_0|^{\gamma}), \quad (k_0 = \text{const.}). \tag{9}$$

So $U(x,t)$ is Lévy stable distribution with the index α. It follows from (8) that $p(x,t)$ describes further the convection of particles by the constant velocity γ in contrast with $U(x,t)$. We focus our attention on the discussion for $U(x,t)$ in the following.

In order to obtain the inverse in (7) we rewrite it as

$$U(x,t) = \frac{1}{\pi}\mathcal{F}_c[e^{-Dt|k|^{\alpha}}, |k| \to |x|], \tag{10}$$

where $\mathcal{F}_c$ denotes the Fourier cosine transform

$$\mathcal{F}_c[g(u), u \to v] = \int_0^{\infty} g(u)\cos uv du. \tag{11}$$

Using the Fox function representation (Mathai and Saxena, 1978; Srivastava et al., 1982; Duan, 2005)

$$e^{-Dt|k|^{\alpha}} = \frac{1}{\alpha}H_{0,1}^{1,0}\left((Dt)^{1/\alpha}|k|\big|_{(0,1/\alpha)}\right) \tag{12}$$

and the Fourier cosine transform of Fox functions (Glöckle and Nonnenmacher, 1993)

$$\mathcal{F}_c[H_{p,q}^{m,n}(z), z \to v] = \frac{\pi}{v}H_{q+1,p+2}^{n+1,m}\left(v\left|\begin{matrix}(1-b_j,\beta_j),(1,1/2)\\(1,1),(1-a_j,\alpha_j),(1,1/2)\end{matrix}\right.\right), \quad \mu \le 1, \tag{13}$$

$$\mathcal{F}_c[H_{p,q}^{m,n}(z), z \to v] = \frac{\pi}{v}H_{p+2,q+1}^{m,n+1}\left(\frac{1}{v}\left|\begin{matrix}(0,1),(a_j,\alpha_j),(0,1/2)\\(b_j,\beta_j),(0,1/2)\end{matrix}\right.\right), \quad \mu \geqslant 1, \tag{14}$$

where $\mu = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j$, we obtain from Eq. (10)

$$U(x,t) = \frac{1}{\alpha|x|}H_{2,2}^{1,1}\left(\frac{(Dt)^{1/\alpha}}{|x|}\left|\begin{matrix}(0,1),(0,1/2)\\(0,1/\alpha),(0,1/2)\end{matrix}\right.\right), \quad 0 < \alpha \le 1,$$

$$U(x,t) = \frac{1}{\alpha|x|}H_{2,2}^{1,1}\left(\frac{|x|}{(Dt)^{1/\alpha}}\left|\begin{matrix}(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right.\right), \quad 1 \leqslant \alpha \le 2. \tag{15}$$

As $\alpha = 1$, the first expression in Eq. (15) permits $|Dt/x| < 1$, while the second permits $|Dt/x| > 1$. Making use of the series expression of Fox functions and the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

we get the series representations

$$U(x,t) = \frac{1}{\pi|x|}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\sin\frac{\pi\alpha n}{2}\Gamma(1+\alpha n)\left(\frac{Dt}{|x|^{\alpha}}\right)^n, 0 < \alpha \le 1, \tag{16}$$

$$U(x,t) = \frac{1}{\pi\alpha(Dt)^{1/\alpha}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\Gamma\left(\frac{1+2n}{\alpha}\right)\left(\frac{|x|}{(Dt)^{1/\alpha}}\right)^{2n}, 1 \leqslant \alpha \le 2. \tag{17}$$

3. Discussions and conclusions

As $\alpha = 1$, the Cauchy distribution is obtained from Eqs. (16) and (17)

$$U(x,t)|_{\alpha=1} = \frac{Dt}{\pi(x^2 + (Dt)^2)}. \tag{18}$$

As $\alpha = 2$, with the help of the identity

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{4^n n!}$$

and the expression in Eq. (17), the Gauss distribution is obtained

$$U(x,t)|_{\alpha=2} = \frac{1}{2\sqrt{\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right). \tag{19}$$

From Eqs. (16) and (17) we have the following asymptotic expressions

$$U(x,t) \sim \sin\frac{\pi\alpha}{2}\Gamma(1+\alpha)\frac{Dt}{\pi|x|^{\alpha+1}}, \quad \frac{Dt}{|x|^{\alpha}} \to 0 \tag{20}$$

for $0 < \alpha \le 1$, and

$$U(x,t) \sim \Gamma\left(\frac{1}{\alpha}\right)\frac{1}{\pi\alpha(Dt)^{1/\alpha}}, \quad \frac{|x|}{(Dt)^{1/\alpha}} \to 0 \tag{21}$$

for $1 \leqslant \alpha \le 2$. Using the asymptotic expansion of Fox functions we get