Journal of King Saud University – Science (2016) **28**, 29–32

King Saud University
Journal of King Saud University – Science
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ORIGINAL ARTICLE

Steady-state response of constant coefficient discrete-time differential systems

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Received 5 January 2015; accepted 19 February 2015
Available online 12 March 2015

KEYWORDS
Fractional differential equations;
Constant coefficient;
Particular solution;
Eigenfunction;
Transfer function

Abstract The problem of steady state output of the discrete-time fractional differential systems is studied in this paper. Based on the fact that the exponentials are the eigenfunctions of such systems, a general algorithm for the output computation when the input is the product "rising factorial. exponential" is presented. The singular case is studied and solved.

1. Introduction

The discrete-time differential systems were studied in Ortigueira et al. (2015) where we developed a framework parallel to the classic used in continuous-time systems. Those systems are based on the nabla and delta derivatives (Bohner and Peterson, 2001; Hilger, 1990; Ortigueira et al., 2015). Here we resume the study of those systems by considering the steady state responses to exponentials and products of exponentials by rising factorial functions. We will study both the regular and singular cases in a way similar to the one followed in Ortigueira (2014b). The algorithm is based on the concept of eigenfunction. As shown in Ortigueira et al. (2015) the eigenfunctions of discrete-time differential systems linear systems are exponentials suitably defined and the corresponding eigenvalues are the transfer functions. Such exponentials are defined with the help of the nabla and delta derivatives and lead to nabla and delta Laplace transforms. We will consider the regular and singular cases; these correspond to the situation of infinite eigenvalue.

The paper outline is as follows. In Section 2 we present the nabla and delta derivatives and the corresponding exponentials. Their properties are listed. In Section 3 we show how to compute the output when the input is an exponential or the product of an exponential by a rising factorial function.

Important remark – The formulation we will present although in a discrete-time setup it mimics the continuous-time counterpart. This leads us to use interchangeably $t = nh$ where h is the underlying time interval.

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Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2015.02.006
1018-3647

2. Fractional nabla and delta derivatives and exponentials

Let $t = nh$ be any generic point in $\mathbf{T} = h\mathbf{Z} = \{kh : k \in \mathbf{Z}\}$. We define the **nabla derivative** (Bohner and Peterson, 2001; Hilger, 1990) by:

$$D_{\nabla}f(t) = f'_{\nabla}(t) := \frac{f(t) - f(t-h)}{h} \tag{1}$$

and the **delta derivative** (Neuman, 1993) by

$$D_{\Delta}f(t) = f'_{\Delta}(t) := \frac{f(t+h) - f(t)}{h} \tag{2}$$

As it can be seen the first one is *causal*, while the second is *anti-causal*. Their generalizations for any real (or complex) order are obtained from the continuous-time Grünwald–Letnikov derivative (Diaz and Osler, 1974; Magin et al., 2011; Ortigueira, 2011; Ortigueira et al., 2015):

$$D^{(\alpha)}_{\nabla}f(t) = f^{(\alpha)}_{\nabla}(t) := \frac{\sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}f(t-nh)}{h^{\alpha}} \tag{3}$$

and

$$D^{(\alpha)}_{\Delta}f(t) = f^{(\alpha)}_{\Delta}(t) := e^{-i\alpha\pi}\,\frac{\sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}f(t+nh)}{h^{\alpha}} \tag{4}$$

As before (Ortigueira, 2011) we will call these derivatives respectively *forward* and *backward* due to the "time flow", from past to future or the reverse. This terminology is the reverse of the one used in some mathematical literature. The first is causal while the second is anti-causal.

Attending to the fact that $(-1)^n\binom{\alpha}{n} = \frac{(-\alpha)_n}{n!}$ where $(-\alpha)_n$ is the Pochhammer symbol for the rising factorial – $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$[2]; we conclude immediately that these derivatives include as special cases the integer order derivatives and anti-derivatives.

These derivatives enjoy several properties as described in Ortigueira et al. (2015). The eigenfunctions of these derivatives are the **nabla and delta generalized exponentials** defined by Ortigueira et al. (2015):

$$e_{\nabla}(t,s) = [1 - sh]^{-t/h} \tag{5}$$

and

$$e_{\Delta}(t,s) = [1 + sh]^{t/h} \tag{6}$$

The properties of these exponentials are described in Ortigueira et al. (2015).

3. Outputs of differential discrete-time linear systems

3.1. Regular cases

3.1.1. Exponential input

We are going to consider systems with the general format (Magin et al., 2011)

$$\sum_{k=0}^{N} a_k D^{\alpha_k} y(t) = \sum_{k=0}^{M} b_k D^{\beta_k} x(t) \tag{7}$$

with $a_N = 1$. The operator D is the nabla derivative defined above . The orders N and M are any positive integers. The α_k and β_k sequences are strictly increasing and positive real numbers.

The discrete-time convolution between two discrete-time functions $f(t)$ and $g(t)$ is given by:

$$f(t) * g(t) = h\sum_{k=-\infty}^{+\infty} f(kh)g(nh - kh) \tag{8}$$

Introduce the discrete delta (impulse) function by:

$$\delta(nh) = D\varepsilon(nh) \tag{9}$$

where $\varepsilon(nh)$ is the discrete-time Heaviside unit step

$$\varepsilon(nh) = \begin{cases} 1 & n \geqslant 0 \\ 0 & n < 0 \end{cases} \tag{10}$$

Let $g(t)$ be the impulse response of the system defined by (7): $x(t) = \delta(nh)$. The output is the convolution of the input and the impulse response (Ortigueira et al., 2015).

$$y(t) = g(t) * x(t) \tag{11}$$

If $x(t) = e_{\nabla}(nh, s)$ the output is given by:

$$y(t) = e_{\nabla}(nh,s)\left[h\sum_{n=-\infty}^{\infty} g(nh)e_{\Delta}(nh,-s)\right]$$

The summation expression will be called **transfer function** as usually. We write then

$$G(s) = h\sum_{n=-\infty}^{\infty} g(nh)e_{\Delta}(nh,-s) \tag{12}$$

say, the transfer function is the nabla Laplace transform (Ortigueira et al., 2015) of the impulse response. It is important to remark that the nabla Laplace transform uses the delta exponential. There is also the delta Laplace transform (see Ortigueira et al., 2015). With these results we can easily express the transfer function as

$$G(s) = \frac{\sum_{k=0}^{M} b_k s^{\beta_k}}{\sum_{k=0}^{N} a_k s^{\alpha_k}} \tag{13}$$

We conclude that:

- The exponentials are the eigenfunctions of the linear systems (7)
- The eigenvalues are the transfer function values.

Putting $s = \frac{1-e^{i\theta}}{h}$ we obtain the usual sinusoidal case. These results exhibit a high degree of coherence with classic results (Ortigueira, 2014a).

Example 1. Let $h = 1$ and consider the differential equation (Ortigueira, 2014a)

$$y'''(t) + y''(t) - 4y'(t) + 2y(t) = x(t)$$

Let $x(n) = 2^{-n}$. This corresponds to $s = -1$. The solution is given by:

$$y(n) = \frac{1}{(-1)^3 + (-1)^2 - 8 + 2}2^{-n} = \frac{1}{6}2^{-n}$$

The above result can be generalized.

[2] We make the convention $(0)_0 = 1$ and $(0)_n = 0$ for any integer n