Nonlinear Analysis: Real World Applications 34 (2017) 61–74

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Weak solutions to the two-dimensional steady compressible Navier–Stokes equations

Xin Zhong*

School of Mathematics and Statistics, Southwest University, Chongqing 400715, PR China

A R T I C L E I N F O

Article history:
Received 25 June 2015
Received in revised form 3 August 2016
Accepted 5 August 2016

Keywords:
Steady fluids
Compressible Navier–Stokes system
Viscous isothermal fluid
Weak solution

A B S T R A C T

We study motions of the steady compressible viscous isothermal fluids in a bounded two dimensional domain governed by the Navier–Stokes equations. We obtain that there exists at least one weak solution to such a problem satisfying some integrability. The proof of this result is based on delicate estimates for the momentum equations and a bootstrapping argument.

1. Introduction

We consider the Navier–Stokes equations for compressible fluids in the steady isothermal case in a bounded domain $\Omega \subset \mathbb{R}^2$:

$$\operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$-\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla \operatorname{div}\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P(\rho) = \rho\mathbf{f}, \tag{1.2}$$

where $\rho \geq 0$, $\mathbf{u} = (u^1, u^2)$ and $P(\rho) = A\rho$ with a positive constant $A > 0$ are the fluid density, velocity, and pressure, respectively. The constant viscosity coefficients μ and λ satisfy

$$\mu > 0, \qquad \lambda + \mu > 0. \tag{1.3}$$

$\mathbf{f}$ is a given vector field which models an outer force density.

Eqs. (1.1)–(1.2) are supplemented with vorticity free boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad \partial_{x_2}u_1 = \partial_{x_1}u_2 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\mathbf{n}$ denotes the outer normal to the boundary $\partial\Omega$.

* Fax: +86 10 82541345.
E-mail address: xzhong1014@amss.ac.cn.

http://dx.doi.org/10.1016/j.nonrwa.2016.08.004

Moreover, the total mass is prescribed

$$\int_{\Omega} \rho(x)dx = M > 0. \tag{1.5}$$

Before stating our main results, let us introduce some notations in this paper. As usual, when we establish a priori estimates, a certain number of constants appear in the inequalities. However, so as not to make the notations too cumbersome, these constants are always denoted as C even though their value may possibly change from one line to another. The dependence of these constants on various parameters of the problem is only referred to when necessary. Vector or tensor fields will be denoted with boldface characters. Moreover, in what follows, the convention of summation over repeated indices is used.

For functional spaces, $L^p(\Omega), 1 \le p \le \infty$, denotes the usual Lebesgue space on Ω and $\|\cdot\|_p$ denotes its L^p norm. $W^{k,p}(\Omega)$ denotes the standard kth order Sobolev space and $\|\cdot\|_{k,p}$ denotes $W^{k,p}$ norm. To simplify the notation, if not stated explicitly otherwise, we do not distinguish between the spaces of vector and scalar valued functions; e.g. both $L^p(\Omega)$ and $L^p(\Omega;\mathbb{R}^2)$ are denoted $L^p(\Omega)$. The difference is always clear from the context. $X(\Omega;\mathbb{R}^2)$ (resp. $X_{\mathbf{n}}(\Omega;\mathbb{R}^2)$) stands for the Banach space of vector fields with component belonging to $X(\Omega)$ (resp. with zero normal trace at the boundary Ω). We also use $\|\cdot\|_X$ to denote $X(\Omega;\mathbb{R}^2)$ norm. $L^{p,\alpha}(\Omega)$ $(0 < \alpha \le 2)$ denotes the Morrey spaces over Ω, i.e., those $q(x)$ for which

$$\sup_{r>0, y\in\Omega} \left(r^{\alpha-2} \int_{\Omega\cap B_r(y)} |q(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

where $B_r(y)$ is a ball centered at y with radius $r > 0$. When $p = 1$, we allow $q(x)$ to be a measure.

We will use the following notion of the weak solutions to the steady Navier–Stokes equations (1.1)–(1.2).

Definition 1.1. By a weak solution of the system (1.1)–(1.5) we mean a pair $(\rho, \mathbf{u}) \in L^{1+\varepsilon}(\Omega) \times W^{1,2}_{\mathbf{n}}(\Omega)$ for some $\varepsilon > 0$ such that

- $\rho \ge 0$ a.e. in Ω, $\int_\Omega \rho(x)dx = M$.
- For every $\varphi \in C^1(\overline{\Omega})$, it holds that

$$\int_\Omega \rho\mathbf{u}\cdot\nabla\varphi dx = 0. \tag{1.6}$$

- For all $\boldsymbol{\varphi} \in C^1_{\mathbf{n}}(\Omega)$, there holds

$$\mu\int_\Omega \nabla\mathbf{u} : \nabla\boldsymbol{\varphi} dx + (\lambda+\mu)\int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\boldsymbol{\varphi} dx = A\int_\Omega \rho\operatorname{div}\boldsymbol{\varphi} dx + \int_\Omega \rho\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\varphi} dx + \int_\Omega \rho\mathbf{f}\cdot\boldsymbol{\varphi} dx. \tag{1.7}$$

For the existing mathematical results concerning weak solutions of two-dimensional steady compressible Navier–Stokes equations for large data, the first breakthrough into the existence of weak solutions to steady compressible Navier–Stokes equations is made in the Lions' book [1], where Lions proved existence of weak solutions to the N-dimensional steady flow under some suitable assumptions. It is worth noticing that, in two dimensions, Lions considered the case of the pressure law $P(\rho) = A\rho^\gamma$ with $\gamma > 1$. Concerning $\gamma = 1$, Lions can only handle weak solvability with a slight modification of (1.1) $a\rho + \operatorname{div}(\rho\mathbf{u}) = 0$ for $a > 0$. In the isothermal flow situation, i.e., $\gamma = 1$, the pressure term in the momentum equation does not cause any trouble. All difficulties are related to the convective term $\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})$ because of the lack of the higher integrability of the density ρ in $L^p(\Omega)$. For more details see [1, Chapter 6]. Later, Plotnikov and Sokolowski [2]