Nonlinear Analysis: Real World Applications 32 (2016) 91–97

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

# Non-integrability of the Karabut system

Ognyan Christov

*Department of Mathematics and Informatics, Sofia University, 5 J. Bouchier blvd., 1164, Sofia, Bulgaria*

ARTICLE INFO

*Article history:*
Received 1 March 2016
Accepted 6 April 2016
Available online 5 May 2016

*Keywords:*
Solitary wave in a fluid
Witting series
Karabut system
Non-Hamiltonian integrability
Differential Galois groups

ABSTRACT

In order to characterize the solitary wave in a fluid of finite depth, Witting introduced a specific power series (the Witting series). Karabut demonstrated that the problem of summation of the Witting series is brought to the integration of a particular system of ordinary differential equations and solved this system in the cases when the number of the equations is three or four. We give a simple proof that the Karabut system of five equations is already non-integrable in non-Hamiltonian sense using the Differential Galois approach.



## 1. Introduction

We study a plane vortex-free stationary flow of an ideal incompressible heavy fluid over a flat bottom. Detailed formulation of the problem can be found in Karabut [1–3] and we follow mainly his description.

Let $(X, Y)$ be a Cartesian coordinate system with $X$-axis aligned along the bottom, $h_0$ be the depth of the unperturbed fluid at infinity and $u_0$ be the velocity of the flow at infinity.

The problem of constructing a solitary wave is reduced to the finding of a solution in the form $Y = Y_0(X)$, which fulfils the condition $\lim_{|X|\to\infty} Y_0(X) = h_0$. This task hinges on a single parameter. Usually the Stokes parameter $\theta$ ($0 \le \theta < \pi/2$) is taken which in its turn is related with the Froude number ($Fr = u_0/\sqrt{gh_0} > 1$) via $\tan\theta/\theta = Fr^2$.

Denote by $\Phi$ the velocity potential and by $\Psi$ the streamline function. In the plane of the non-dimensional complex potential $\chi = \varphi + i\psi = \theta(\Phi + \Psi)/h_0u_0$, the strip

$$-\infty < \varphi < \infty \quad (0 < \psi < \theta) \tag{1}$$

is in correspondence with the fluid. The solitary-wave problem will be solved if we obtain the conformal map of this strip onto the flow area. This map can be written as $Z = X + iY = \frac{h_0}{\theta}(\chi + W(\chi))$. Here the function

*E-mail address:* christov@fmi.uni-sofia.bg.

http://dx.doi.org/10.1016/j.nonrwa.2016.04.002
1468-1218/

$W(\chi)$ is a solution of the following boundary-value problem

$$\left|\frac{d(W+\chi)}{d\chi}\right|^2 = \frac{1}{1-2\nu \operatorname{Im} W}, \qquad \nu = \cot\theta \ (\psi=\theta,\ \varphi<\varphi_0), \tag{2}$$

$$\operatorname{Im} W = 0, \qquad (\psi = 0,\ \varphi<\varphi_0), \tag{3}$$

$$\lim_{\varphi\to-\infty} \operatorname{Im} W = 0. \tag{4}$$

Note that among the solutions of (2)–(4), the solitary-wave is subjected to the condition

$$\varphi_0 = +\infty, \qquad \lim_{\varphi\to\infty} \operatorname{Im} W = 0. \tag{5}$$

In a number of earlier papers the solution of the problem of finding a solitary wave is represented by series of different types (see [4,1–3]).

For instance, if we are looking for a solution of (2)–(4) presented as a series of the kind

$$W = \sum_{j=1}^{\infty} \theta^{2j} W^j(\chi) \tag{6}$$

this yields the shallow-water expansion. It occurs that the functions $W^j(\chi)$ can be given as polynomials of $\cosh^{-2}(\frac{\chi}{2})$. Then it is reasonable to put $\zeta = e^{\chi}$ and to rewrite (6) as

$$W = \sum_{j=1}^{\infty} E_j(\theta)\zeta^j, \quad \operatorname{Im} E_j = 0. \tag{7}$$

This type of series was suggested by Witting [4]. One can easily obtain recurrent formulas for the coefficients $E_j$: $E_1$ can be any positive number.

This series has been investigated numerically for $\theta = \pi/3$ and $\theta = \pi/4$ in [4]. Karabut [1–3] has shown that for $\theta = m\pi/n$, where $m$ and $n$ are integers, the problem of exact summation of the Witting series is equivalent to the solution of a special system of $n$ ordinary differential equations. The following functions are introduced

$$P_j(\chi) = W(\zeta\omega^{2j-2}), \quad \omega = e^{i\theta},\ j = 1,\dots,n \tag{8}$$

and it turns out that they satisfy the following system

$$\left(\frac{dP_{j+1}}{d\chi}+1\right)\left(\frac{dP_j}{d\chi}+1\right) = \frac{1}{f_j}, \qquad P_{n+1}\equiv P_1, \quad j=1,\dots,n, \tag{9}$$

where $f_j = 1 + i\nu(P_{j+1} - P_j)$. Therefore, to deal with the boundary-value problem (2)–(4) in the form of the Witting series (7), it is enough to integrate the system (9) and to take $W = P_1$.

The system (9) has been integrated for $\theta = \pi/3$ in [1] and for $\theta = \pi/4$ in [3], that is, in these cases the Witting series are summed up exactly.

Here, we consider the case $\theta = \pi/5$ when (9) contains five equations. In this case (and more generally, for all $\theta = \pi/n$, $n$ is an odd integer), the system (9) can be written in the standard form. Denote $\triangle = \sqrt{f_1 f_2 f_3 f_4 f_5}$. Then Eq. (9) can be written with respect to the variables $f_j$ in the following way:

$$\frac{df_1}{d\chi} = i\nu\frac{f_3f_5 - f_2f_4}{\triangle}, \qquad \frac{df_2}{d\chi} = i\nu\frac{f_4f_1 - f_3f_5}{\triangle}, \qquad \frac{df_3}{d\chi} = i\nu\frac{f_5f_2 - f_4f_1}{\triangle},$$
$$\frac{df_4}{d\chi} = i\nu\frac{f_1f_3 - f_5f_2}{\triangle}, \qquad \frac{df_5}{d\chi} = i\nu\frac{f_2f_4 - f_1f_3}{\triangle}. \tag{10}$$