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On the existence time of local solutions for critical semilinear Schrödinger equations in Sobolev spaces

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The existence time of local solutions of semilinear Schrödinger equations in Sobolev spaces is considered based on the method of frequency decomposition. The semilinear terms are power type or exponential type, which are critical in terms of the scaling or the Trudinger–Moser inequality. The existence time is estimated by the low and high frequency parts of data.

1. Introduction

The main aim of this paper is to consider the semilinear terms of exponential type (see Theorem 1.4). However, we start from the semilinear terms of polynomial type to describe the background of our problem. Let us consider the Cauchy problem for the semilinear Schrödinger equation

$$\begin{cases} i\dfrac{\partial u}{\partial t}(t,x) + \Delta u(t,x) + \kappa(|u|^{p-1}u)(t,x) = 0 \quad \text{for } (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(0,\cdot) = u_0(\cdot) \in H^s(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where $i := \sqrt{-1}$, $T > 0$, $n \geq 1$, $0 \leq s < n/2$, $p = p(s) := 1 + 4/(n-2s)$, $\kappa \in \mathbb{C}$, $\Delta := \sum_{j=1}^{n} \partial^2/\partial x_j^2$ is the Laplacian, u_0 is a given initial datum, and $H^s(\mathbb{R}^n)$ denotes the Sobolev space of order s. The number $p(s)$ is known as the scaling critical number for the Cauchy problem. Namely, the first equation in (1.1), and the norm $\|u\|_{\dot{H}^s(\mathbb{R}^n)}$ are both invariant by the scaling $u_\lambda(t,x) := \lambda^{2/(p-1)} u(\lambda^2 t, \lambda x)$ for any $\lambda \in \mathbb{R}$ with $\lambda \neq 0$, where $\dot{H}^s(\mathbb{R}^n)$ denotes the homogeneous Sobolev space of order s. For the case $1 < p \leq p(s)$, it is known that the solution of (1.1) exists if T and $\|u_0\|_{\dot{H}^s(\mathbb{R}^n)}$ satisfy

$$T^{(p(s)-p)/(p(s)-1)} \|u_0\|_{\dot{H}^s(\mathbb{R}^n)}^{p-1} \leq C \tag{1.2}$$

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for some constant $C > 0$ which is independent of T and u_0. So that, we are able to take T as

$$T = T(\|u_0\|_{\dot{H}^s(\mathbb{R}^n)}) = \left(\frac{C}{\|u_0\|^{p-1}_{\dot{H}^s(\mathbb{R}^n)}} \right)^{(p(s)-1)/(p(s)-p)}$$

when p is subcritical $p < p(s)$. Therefore, T is estimated from below dependent only on the norm $\|u_0\|_{\dot{H}^s(\mathbb{R}^n)}$. When p is critical $p = p(s)$, the condition (1.2) is written as $\|u_0\|^{p-1}_{\dot{H}^s(\mathbb{R}^n)} \le C$. Thus, u is a global solution (i.e., $T = \infty$) if $\|u_0\|_{\dot{H}^s(\mathbb{R}^n)}$ is sufficiently small (see [1,2]). For large $\|u_0\|_{\dot{H}^s(\mathbb{R}^n)}$ when $p = p(s)$, the existence time T depends on the profile of the datum u_0 in addition to the norm $\|u_0\|_{\dot{H}^s(\mathbb{R}^n)}$. For example, it is known that the solution u for large u_0 exists if $\|e^{it\Delta}u_0\|_{L^q((0,T),\dot{B}^s_{r,2}(\mathbb{R}^n))}$ is sufficiently small for some admissible pair (q, r) with $q < \infty$ (see e.g., [3, Theorem 1.2]). Although we are able to take such T since $q < \infty$, the dependency of T on u_0 is not clear and it has not been characterized. This situation yields a difficulty to consider global solutions for large data for the critical case $p = p(s)$ since the dependency of T on the norms of data are used to show global solutions from local solutions by conservation laws. The aim of this paper is to show that T can be characterized by the norms of low and high frequency parts of data.

To state our theorems, we set some function spaces as follows. We say that the pair (q, r) is admissible if q and r satisfy $2 \le q, r \le \infty$, $1/r + 2/nq = 1/2$, and $(q, r, n) \ne (2, \infty, 2)$. For given admissible pairs $\{(q_j, r_j)\}_{j=1}^2$, we put

$$X^s(T) := L^\infty((0,T), H^s(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,T), B^s_{r_j,2}(\mathbb{R}^n)),$$

where $B^s_{r_j,2}(\mathbb{R}^n)$ denotes the Besov space. Let φ be a smooth and nonnegative function on $\mathbb{R}^n$ such that its Fourier transform $\mathcal{F}\varphi$ satisfies $\operatorname{supp} \mathcal{F}\varphi \subset \{\xi \in \mathbb{R}^n \mid 1/2 \le |\xi| \le 2\}$, and $\sum_{j=-\infty}^{\infty} (\mathcal{F}\varphi)(\xi/2^j) = 1$ for any $\xi (\ne 0) \in \mathbb{R}^n$. We put $\varphi_j := \mathcal{F}^{-1}((\mathcal{F}\varphi)(\cdot/2^j))$ for $j \in \mathbb{Z}$, $\psi := \mathcal{F}^{-1}(1 - \sum_{j=1}^{\infty} \mathcal{F}\varphi_j)$. The Besov space $B^s_{r,2}(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $1 \le r \le \infty$ is defined by

$$B^s_{r,2}(\mathbb{R}^n) := \left\{ v \in \mathcal{S}'(\mathbb{R}^n) \mid \|v\|_{B^s_{r,2}(\mathbb{R}^n)} < \infty \right\},$$

where we have put

$$\|v\|_{B^s_{r,2}(\mathbb{R}^n)} := \|\psi * v\|_{L^r(\mathbb{R}^n)} + \left\{ \sum_{j=1}^{\infty} 2^{2sj} \|\varphi_j * v\|^2_{L^r(\mathbb{R}^n)} \right\}^{1/2},$$

$\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions and $*$ denotes the convolution by spatial variables. For a natural number $N \ge 1$, we put $\sigma_N := \sum_{j=N}^{\infty} \varphi_j$, and $\chi_N := \mathcal{F}^{-1}(1 - \mathcal{F}\sigma_N)$. We say that $\sigma_N * u$ and $\chi_N * u$ are the high and low frequency parts of u, respectively. For any real number $\delta > 0$ and any datum $u_0 \in H^s(\mathbb{R}^n)$, there exists a natural number N which satisfies $\|\sigma_N * u_0\|_{\dot{H}^s(\mathbb{R}^n)} \le \delta$ since we have $\lim_{N\to\infty} \|\sigma_N * u_0\|_{\dot{H}^s(\mathbb{R}^n)} = 0$ by the definition of σ_N.

We have the following characterization on the existence time T in the critical case $p = p(s)$.

Theorem 1.1. *Let $n \ge 1$, $0 \le s < n/2$, $p = p(s)$, $\kappa \in \mathbb{C}$. Assume $s < p$ when p is not an odd number. There exist two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$ and a real number $\delta > 0$ with the following property. For any $u_0 \in H^s(\mathbb{R}^n)$, let N be a number which satisfies $\|\sigma_N * u_0\|_{\dot{H}^s(\mathbb{R}^n)} \le \delta$. There exist $T > 0$ and a unique solution $u \in C([0,T), H^s(\mathbb{R}^n)) \cap X^s(T)$ of (1.1), where T depends only on $\|\chi_N * u_0\|_{\dot{H}^s(\mathbb{R}^n)}$, $\|\sigma_N * u_0\|_{\dot{H}^s(\mathbb{R}^n)}$ and N, i.e., $T = T(N, \|\chi_N * u_0\|_{\dot{H}^s(\mathbb{R}^n)}, \|\sigma_N * u_0\|_{\dot{H}^s(\mathbb{R}^n)})$. Moreover, the solutions depend on the initial data continuously.*

Remark 1.2. The proof of Theorem 1.1 also shows that we are able to take $T = \infty$ if $\|u_0\|_{\dot{H}^s(\mathbb{R}^n)}$ is sufficiently small. Theorem 1.1 is also valid with $\kappa|u|^{p-1}u$ replaced by $\kappa|u|^p$, provided $s < p$ when p is not an even number.