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The fractional Hartree equation without the Ambrosetti–Rabinowitz condition

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ABSTRACT

We consider a class of pseudo-relativistic Hartree equations in presence of general nonlinearities not satisfying the Ambrosetti–Rabinowitz condition. Using variational methods based on critical point theory, we show the existence of two non trivial signed solutions, one positive and one negative.

1. Introduction

In this paper we deal with a general class of pseudo-relativistic Schrödinger equations with a Hartree nonlinearity. Such equations emerge from the description of pseudorelativistic boson stars (see [1] for a physical derivation of the problem), but also as the mean field limit description of a quantum relativistic Bose gas (see [2,3]). Fröhlich and Lenzmann in [4,5] approached the problems of existence, blowing up and stability of solutions. The problem they studied in [4] took the following form:

$$i\psi_t = \sqrt{-\Delta + m^2}\psi - \left(\frac{1}{|x|} * |\psi|^2\right)\psi \quad \text{in } \mathbb{R}^3. \tag{1.1}$$

Here ψ is a complex valued wave function which describes the quantum status of a particle, while the operator involving the square root represents its relativistic kinetic and rest energies, and reduces to the usual half Laplacian $(-\Delta)^{1/2}$ when $m = 0$. Besides, the term $1/|x|$ inside the convolution product stands for the Newtonian gravitational potential in $\mathbb{R}^3$ and represents repulsive forces among the particles.

In [6], a generalized version of (1.1) is studied, allowing for an additional potential term $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, which takes into account other external forces, and, in addition, a general field potential W replaces the Newtonian one. In this setting, Eq. (1.1) takes the following form:

$$i\psi_t = \sqrt{-\Delta + m^2}\psi - \lambda\left(W * |\psi|^2\right)\psi - f(x, \psi) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

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with $\lambda \in \mathbb{R}$.

In this paper we search solutions of a problem corresponding to (1.2) but settled in a bounded domain Ω of $\mathbb{R}^N$. This allows us to remove the hypothesis of radial symmetry of the solutions and of the potential f assumed in [6]. The problem we study is the following one:

$$\begin{cases} i\psi_t(x,t) = \sqrt{-\Delta + m^2}\psi(x,t) - \lambda\left(\int_\Omega G(x,y)|\psi(y,t)|^2 dy\right)\psi(x,t) - f(x,\psi(x,t)) & \text{in } \Omega, \\ \psi(x,t) \quad = 0 & \text{on } \partial\Omega,\ \forall t \end{cases} \tag{1.3}$$

with $\lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^N$ bounded.

In (1.3), passing from $\mathbb{R}^N$ to Ω, we have replaced the Newtonian-like kernel $W(x-y)$ with the Green function $G(x,y)$ associated to the Laplace operator in Ω (indeed this is the Coulomb-type interaction between particles in boson stars), and we consider homogeneous boundary conditions on $\partial\Omega$. In this way, the corresponding potential ϕ at time t takes the form $\phi(x) = \int_\Omega G(x,y)|\psi(y,t)|^2 dy$. From now on we will adopt the symbol $\langle G, \psi\rangle = \int_\Omega G(x,y)|\psi(y,t)|^2 dy$ for the previous potential term. We also note that potential $\phi(x) = \langle G, \psi\rangle$ is the solution of the linear problem

$$\begin{cases} -\Delta\phi(x,t) = 4\pi|\psi(x,t)|^2 & \text{in } \Omega \\ \phi = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.4}$$

for every t, so that problem (1.3) can be written as a system with an additional equation for ϕ, as similarly done in [7–10].

It is worth reminding some general properties of Green functions for C^1 bounded domains Ω, which we shall use later (for instance, see [11]). Green functions $G : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$ are non negative, symmetric with respect to their variables, and when $N \geq 3$ they verify the inequality $G(x,y) \leq C|x-y|^{2-N}$, where the right hand side is the kernel of the Newtonian like potential. More generally, inspired by the above inequality, in our setting we consider a function G which is symmetric, non negative and satisfies certain integrability conditions. To be precise, we will require $G(x,y) \leq W(x-y)$, with W satisfying some integrability conditions which cover the case of the Newtonian kernel for $N \geq 3$.

In order to obtain existence of solutions for problem (1.3), it is crucial to specify some hypothesis on the external potential $F(x,\psi) = \int_0^\psi f(x,s)ds$. The prototype for F is a power-like potential, so it is natural to require $F(x,s) = F(x,|s|)$ and $f(x,e^{i\theta}|s|) = e^{i\theta}f(x,|s|)$. This is not restrictive in the setting of Abelian Gauge Theories (see [12–14]), and it allows us to search for real solutions of the stationary equation associated to (1.3). Indeed, we focus on solutions in the form of solitary waves, i.e. on functions of the form

$$\psi(x,t) = e^{-i\omega t}u(x), \tag{1.5}$$

where $\omega \in \mathbb{R}$ and $u : \Omega \to \mathbb{R}$.

After substitution of (1.5) into (1.3), and considering that the operator $\sqrt{-\Delta + m^2}$ acts only on the spatial coordinates, we see that function u satisfies the following stationary equation:

$$\begin{cases} \sqrt{-\Delta + m^2}u - \omega u - \lambda\left\langle G, u^2\right\rangle u - f(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

We remark that a standard assumption on F for solving stationary equations like (1.6) is the fulfillment of the usual Ambrosetti–Rabinowitz condition, see [12,13,15](or a reversed one, see [6]). We recall that this condition reads as follows: there exists $\mu > 2$ such that

$$0 < \mu F(x,s) \leq sf(x,s) \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{1.7}$$

Condition (1.7) is very useful to prove that Palais–Smale sequences are bounded, and in turn that Palais–Smale condition (PS-condition in short) holds, so that an essential ingredient in variational methods is guaranteed.