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A boundary control problem for the blood flow in venous insufficiency. The general case

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ABSTRACT

The purpose of this article is to introduce and study an optimal control problem with medical applications. When a vein loses its elasticity, phenomena such as stagnation and recirculation of the blood may appear; these phenomena produce medical complications. We propose an optimization model in order to diminish the negative consequences of the lack of vein elasticity. We extend a previous model involving the interaction between a viscous fluid and an elastic boundary to the case when both the fluid and the elastic medium occupy three dimensional domains. After establishing the existence and uniqueness of the solution for the coupled problem, we present a boundary control problem in order to determine an exterior compression that realizes a blood flow without recirculation. Since it is not possible to find such a compression directly, we consider a sequence of cost functionals and we study the corresponding optimal control problems. The existence and uniqueness of the optimal controls are proved and the optimality conditions that characterize the optimal controls are derived. Finally, we establish the relation between the control problem with physical meaning and the sequence of optimal controls already constructed.

1. Introduction

The so-called fluid–structure interaction problem that describes a fluid interacting with a moving or deformable structure has many practical applications in acoustics, biology, biomechanics, medical sciences, engineering, etc. For this reason, in the last two decades such problems have been studied extensively from the mathematical point of view. For instance, some articles dealing with theoretical results concerning the existence of solutions when the domain occupied by the fluid is either fixed or is time-dependent are [1–3]. In other articles devoted to this type of problems an asymptotic analysis of the fluid–structure interaction was presented (see e.g. [4–9]). We can also give examples of articles presenting fluid–structure interaction problems with direct practical applications: [10] with applications in acoustics, [11] with applications in angioplasty, [12], [13] with applications in medical sciences are only a few of them. In [14] we considered the fluid–structure interaction model in order to describe the interaction between the blood and the vein walls when the thickness of the walls is neglected. When a vein loses its elasticity, phenomena such as

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stagnation and recirculation of the blood may appear; these phenomena produce medical complications such as leg edema and venous ulcers. We proposed an optimization model in order to diminish the negative consequences of the lack of vein elasticity. The approach was performed for a simplified case, in which we reduced the dimension of the elastic structure (vein walls), considering that the structure is much thinner than the fluid domain. In this way, the elastic medium became a part of the fluid domain boundary and some aspects concerning this medium were disregarded. But in reality the elastic structure has a thickness that determines more complex interactions than an elastic boundary. The purpose of this work is to extend the model involving the interaction between a viscous fluid and an elastic boundary to the case when both the fluid and the elastic medium occupy three dimensional domains.

In Section 2 we present the medical situation that justifies such a mathematical study, we describe the geometry of the interaction problem and we give the coupled system which models the physical problem. This system involves as unknowns the velocity ($\mathbf{v}$), the pressure (p) of the fluid and the displacement ($\mathbf{u}$) of the elastic medium. Since the functions $\mathbf{v}$ and $\mathbf{u}$ satisfy non homogeneous boundary conditions on some parts of the boundaries of the fluid and solid domains, respectively (relations $(2.7)_{6,7}$), we propose a change of unknown functions which leads to a homogeneous problem for a new unknown triplet $(\mathbf{V}, p, \mathbf{U})$. In the next section we give the variational formulation for the homogeneous system, by choosing an appropriate regularity for the data. By means of the variational problem we establish the existence, the uniqueness and the regularity of the pair $(\mathbf{V}, \mathbf{U})$. We introduce next the third component of the unknown triplet (the pressure p), we prove its uniqueness and an L^2 regularity with respect to both variables, x and t. Consequently, the main results of Sections 2, 3 are represented by existence, uniqueness and regularity properties for the triplet $(\mathbf{v}, p, \mathbf{u})$, solution of the coupled system modeling the fluid–structure interaction, obtained by means of the variational analysis of the problem. The main results of this article are obtained in Sections 4 and 5, where we introduce and study the boundary control problem. We want to control the recirculation phenomenon of the blood through an inelastic vein by means of some compression forces that act on the exterior boundary of the elastic medium. With respect to the geometric framework presented in Section 2, a blood flow without recirculation is expressed by means of the mathematical condition

$$\mathbf{v} \cdot \mathbf{e}_3 \geq 0, \tag{1.1}$$

with $\mathbf{v}$ the fluid velocity and $\mathbf{e}_3$ the unit vector of Ox_3 axis. So, minimizing the blood recirculation means to minimize the norm of the negative part of the component of $\mathbf{v}$ with respect to Ox_3 axis. This leads to a choice of the cost functional as below

$$J(\mathbf{g}) = \frac{1}{2} \int_0^T \int_{\Omega_f} (\min(\mathbf{v} \cdot \mathbf{e}_3, 0))^2 \mathrm{d}x\mathrm{d}t, \tag{1.2}$$

with $\mathbf{g}$ representing the exterior compression and $\mathbf{v}$ the first component of the unique solution $(\mathbf{v}, p, \mathbf{u})$ of the physical problem corresponding to $\mathbf{g}$. The cost functional is chosen in order to determine some exterior forces that realize a blood flow without recirculation. Since it is not possible to find such forces directly, we consider a sequence of cost functionals obtained from J by adding the term $\frac{\alpha}{2}\|\mathbf{g}\|^2_{H^2(0,T;(L^2(\Gamma_s))^3)}$, with α a small parameter and we study the optimal control problems associated with the regularized cost functionals. The existence and uniqueness of the optimal controls are obtained and the optimality conditions that characterize the optimal controls are derived. Determining an exterior compression that realizes a flow without recirculation means to determine a function $\mathbf{g}^*$ such that $J(\mathbf{g}^*) = 0$. Finally, we discuss when such a situation occurs with respect to the sequence of optimal controls already constructed.

2. Justification and description of the physical problem

The blood flow through a leg vein has an anti-gravity sense; this sense is modified when the vein loses its elasticity, determining stagnation and recirculation of the blood. For reducing the medical complications,