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Method Article

Multiple-rank modification of symmetric eigenvalue problem

HyungSeon Oh*, Zhe Hu

Electrical Engineering, State University of New York at Buffalo, United States

HIGHLIGHTS

- Efficient method to find the eigenvalue/eigenvector pairs of a multiple rank update.
- Application of the Sturm Theorem for a function in a fractional form.
- Modification of a secular equation to exclude identified eigenvalues.

GRAPHICAL ABSTRACT

ABSTRACT

Rank-1 modifications applied k-times ($k > 1$) often are performed to achieve a rank-k modification. We propose a rank- k modification for enhancing computational efficiency. As the first step toward a rank- k modification, an algorithm to perform a rank-2 modification is proposed and tested. The computation cost of our proposed algorithm is in $O(n^2)$ where n is the cardinality of the matrix of interest. We also propose a general rank-k update algorithm based on the Sturm Theorem, and compare our results to those of the direct eigenvalue decomposition and of a perturbation method.

* Corresponding author.
E-mail address: hso1@buffalo.edu (H. Oh).

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Method details

Optimal power flow plays a key role in the operation and planning studies for power systems. Due to its nonlinearity and non-convexity, a numerical solution is pursued using heuristic methods [1,2]. The most widely used method is a Lagrange relaxation, which relies on computationally expensive matrix factorizations. A matrix commonly involves a low-rank update; therefore, it would be a viable option to update the factors instead of the expensive re-factorization process to evaluate them. An *LU* modification method is first introduced, and many study results show its effectiveness [8,9]. While very relevant to power system studies because it can preserve the sparsity, the lack of numerically stability is an important issue. A stable update is achieved by updating Cholesky factorization [9–11]; however, its applicability is limited to positive semi-definite (PSD) matrices. The matrices associated with power systems are not, in general, PSD. Modifying eigenvalue/eigenvector pairs [12] would be a good candidate as they are numerically stable and can modify a large-scale matrix.

Consider a real symmetric matrix $A \in Rn \times n$ with known eigenvalue decomposition $A = Q\Lambda QT$, to which a symmetric perturbation is added. Q is the matrix comprised of eigenvectors and Λ is a diagonal eigenvalue matrix, i.e., i^{th} column vector q_i of Q is the i^{th} eigenvector, and the i^{th} diagonal element of Λ (Λ_i) is the corresponding eigenvalue. If the perturbation is a rank-one matrix, σvvT where σ and v are a scalar and a vector in $Rn \times 1$, the eigenvalue of new matrix $A + \sigma vv^T$ could be given by a “secular equation” [4]:

$$f(\lambda) = 1 + \sigma \sum_{j=1}^{n} \frac{\zeta_j^2}{\lambda_j - \lambda} = 0$$

where λi is the ith eigenvalue of original matrix A, and ζ_i is the ith element in vector z = QTv. If ζi are all nonzero and λi are distinct, then this equation has n solutions. On interval $[\lambda_i, \lambda_{i+1}]$, function f is monotonic. These properties lead to several efficient and stable methods to solve the secular equation [4,5].

However, it is difficult to find the rank-1 modification matrix during the heuristic algorithm. In most situations, the perturbation is not a rank-1 matrix. A reduction of the rank in the perturbation matrix is possible by utilizing eigenvalue decomposition after selecting a subset of eigen-pairs. The disadvantages of this approach are: 1) computationally expensive eigenvalue decomposition, and 2) poor accuracy when a subset of eigenvalue pairs are included. It will be beneficial to modify high-rank eigenvalues directly. Eigenvalue decomposition is performed for a symmetric matrix, and the perturbation is made in preserving the symmetry. In an iterative method, binding constraint sets change, which involves an update such as $e_j^T a + a^T e_j$ where a is a vector in $Rn \times 1$ and e_j is the j^{th} column vector of the identity matrix in $Rn \times n$. Such a change is easily recognized without any further analysis to identify the perturbation vector as a rank-1 update process. This is the primary motivation for a rank-2 update.

In this paper, we propose a new, direct rank-k modification where $\sqrt{n} \le k \ll n$, i.e., k is small but not negligible such as $k = 1$ or 2. In Section Theory of eigenvalue updates, we present the theoretical background for the multiple-rank modification for rank-k updates. Section Numerical result lists the numerical results, and Section Conclusion outlines conclusions and future works.