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Fefferman-Stein decomposition for Q-spaces and micro-local quantities

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ABSTRACT

In this paper, we study the Fefferman-Stein decomposition of $Q_\alpha(\mathbb{R}^n)$ and give an affirmative answer to an open problem posed by Essén et al. (2000). One of our main methods is to characterize the structure of the predual of $Q_\alpha(\mathbb{R}^n)$ by the micro-local quantities. This result indicates that the norm of the predual space of $Q_\alpha(\mathbb{R}^n)$ depends on the micro-local structure in a self-correlation way.

1. Introduction

In this paper, we give a wavelet characterization of the predual of Q-space $Q_\alpha(\mathbb{R}^n)$ without using a family of Borel measures. By this result, we obtain a Fefferman–Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$. Let R_0 be the unit operator and $R_i, i = 1, \dots, n$, be the Riesz transforms, respectively. In 1972, in the celebrated paper [5], C. Fefferman and E. M. Stein proved the following result.

Theorem 1 *([18], Theorem B)*. *If $f \in BMO(\mathbb{R}^n)$, then there exist $g_0, \dots, g_n \in L^\infty(\mathbb{R}^n)$ such that, modulo constants, $f = \sum_{j=0}^n R_i g_i$ and $\sum_{j=0}^n \|g_j\|_{L^\infty} \le C\|f\|_{BMO}$.*

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The importance of the Fefferman–Stein decomposition lays in two aspects. On the one hand, there is a close relation between the $\bar{\partial}$-equation and the Fefferman–Stein decomposition. On the other hand, this decomposition helps understand better the structure of $BMO(\mathbb{R}^n)$ and the distance between $L^\infty(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. Due to the mentioned two points, the Fefferman–Stein decomposition of $BMO(\mathbb{R}^n)$ has been studied extensively by many researchers since 1970s. We refer the reader to Jones [6,7] and Uchiyama [18] for further information. In the latest decades, the Fefferman–Stein decomposition is also extended to other function spaces, for example, BLO, C^0 and VMO, see [1,14] and the references therein.

As an analogy of $BMO(\mathbb{R}^n)$, Q-spaces own a similar structure and many common properties. It is natural to seek for a Fefferman–Stien type decomposition of Q-spaces. For the Q-spaces on the unit disk, Nicolau–Xiao [12] obtained a decomposition of $Q_p(\partial\mathbb{D})$ similar to the Fefferman–Stein's result for $BMO(\partial\mathbb{D})$ (see [12], Theorem 1.2). On Euclidean space $\mathbb{R}^n$, Essen–Janson–Peng–Xiao [4] introduced $Q_\alpha(\mathbb{R}^n)$ as a generalization of $Q_p(\partial\mathbb{D})$. For $\alpha \in (-\infty, \infty)$, $Q_\alpha(\mathbb{R}^n)$ is defined as the space of all the measurable functions with

$$\sup_I |I|^{2\alpha/n-1} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}} dxdy < \infty, \tag{1.1}$$

where the supremum is taken over all cubes I with the edges parallel to the coordinate. They studied $Q_\alpha(\mathbb{R}^n)$ systemically and listed the Fefferman–Stein decomposition of $Q_\alpha(\mathbb{R}^n)$ as one of the open problems.

Problem 1.1 (*[4, Problem 8.3]*)**.** For $n \geq 2$ and $\alpha \in (0, 1)$. Give a Fefferman–Stein type decomposition for $Q_\alpha(\mathbb{R}^n)$.

In this paper, we will give an affirmative answer to this open problem. Generally speaking, there are two methods to obtain the Fefferman–Stein decomposition of $BMO(\mathbb{R}^n)$. The method of Fefferman–Stein [5] is to split BMO-functions involving an extension theorem based on the Hahn–Banach theorem. In [18], A. Uchiyama gave a constructive proof of Theorem 1. In this paper, using wavelets, we study the micro-local structure of $P^\alpha(\mathbb{R}^n)$, which is the predual of $Q_\alpha(\mathbb{R}^n)$. As an application, we obtain a Fefferman–Stein type decomposition of $Q_\alpha(\mathbb{R}^n)$.

For the Fefferman–Stein decomposition of $Q_\alpha(\mathbb{R}^n)$, the difficulties are two-fold.

(1) For a function f in $P^\alpha(\mathbb{R}^n), 0 < \alpha < n/2$, the higher frequency part and the lower frequency part make different contributions to the norm $\|f\|_{P^\alpha}$. That is to say, each $P^\alpha(\mathbb{R}^n)$ has special micro-local structure. As far as we know, there are little results on such structure of $P^\alpha(\mathbb{R}^n)$.
(2) For any function f, the Riesz transforms may cause a perturbation on all the range of its frequencies. To obtain the Fefferman–Stein type decomposition, we need to control the range of the perturbation.

To overcome the above two difficulties, on the one hand, we analyze the micro-local structure of functions in $P^\alpha(\mathbb{R}^n)$. Such micro-local structure can help get a wavelet characterization of $P^\alpha(\mathbb{R}^n)$ without involving a group of Borel measures. On the other hand, we use the classical Meyer wavelets to control the range of the perturbation.

In Section 2, we will give the definition of wavelet basis $\{\Phi^\epsilon_{j,k}\}_{(\epsilon,j,k)\in\Lambda_n}$. It is well-known that a function g can be written as a sum

$$g(x) = \sum_j g_j(x), \quad \text{where } g_j(x) = \sum_{\epsilon,k} g^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x).$$

Let g be a function in Besov spaces or Triebel–Lizorkin spaces. Roughly speaking, the norms of g can be determined by the $l^p(L^q)$-norm or $L^p(l^q)$-norm of $\{g_j\}$, respectively (see [10,17]). For $g \in P^\alpha(\mathbb{R}^n), 0 < \alpha < \frac{n}{2}$, the situation becomes complicated and we cannot use the above approaches. In Section 3, we introduce the micro-local quantities with levels to study the structure of functions in $P^\alpha(\mathbb{R}^n)$.