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Higher integrability of iterated operators on differential forms

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ABSTRACT

In this paper, we first prove the local higher integrability and higher order imbedding theorems for the iterated operators defined on differential forms. Then, we prove the global higher integrability and higher order imbedding inequalities for these operators. Finally, we demonstrate applications of the main results by examples.

1. Introduction

The main purpose of this paper is to prove the higher integrability and higher order imbedding theorems of the iterated operators $D^kG^k(u)$ and $D^{k+1}G^k(u)$, where k is any positive integer, G is Green's operator and $D = d + d^\star$ is the Hodge–Dirac operator applied to differential forms. Here d is the exterior differential operator and $d^\star$ is the Hodge codifferential that is formal adjoint operator of d. Specifically, we prove that both composite operators $D^kG^k(u)$ and $D^{k+1}G^k(u)$ are of higher integrability than that of u. We also obtain the upper bound estimates for the L^s-norms of $D^kG^k(u)$ and $D^{k+1}G^k(u)$ in terms of the L^p-norms of u, where the positive integral exponent s could be much larger than the positive integral exponent p. For the case $k = 1$, the operator $D^{k+1}G^k(u)$ reduces to the composition $D^2G(u) = \Delta G(u)$ of the Laplace–Beltrami operator $\Delta = dd^\star + d^\star d$ and the Green's operator G, which is used to define the well-known Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator. The integrability of operators

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and the upper bound estimates for these operators and their compositions are very important and complicated in the L^p-theory of operators on differential forms. We all know that both the Dirac operator D and Green's operator G are very important operators. These operators are widely studied and very well used in many areas of mathematics and physics, see [1,4,9,11,17,21,13,15,8,18]. Since it was initiated by Paul Dirac in order to get a form of quantum theory compatible with special relativity, the Dirac operator has been playing a critical role in some fields of mathematics and physics, such as quantum mechanics, Clifford analysis and partial differential equations. Our local higher integrability results are presented and proved in Theorems 2.6 and 2.7 and global higher integrability results are presented and proved in Theorems 4.2 and 4.3, respectively. In the meantime, our local higher order imbedding inequalities are presented and proved in Theorems 3.1 and 3.2 and the global ones are stated and showed in Theorems 4.4 and 4.5.

We keep using the traditional notations appearing in [1]. Generally speaking, differential forms are extensions of differentiable functions in $\mathbb{R}^n$. A function $u(x_1, x_2, \dots, x_n)$ is called a 0-form. A differential k-form $u(x)$ is generated by $\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\}$, $k = 1, 2, \dots, n$, that is,

$$u(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

where $I = (i_1, i_2, \dots, i_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$. Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded domain, B and σB be the balls with the same center and $\mathrm{diam}(\sigma B) = \sigma \mathrm{diam}(B)$. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, $\wedge = \wedge(\mathbb{R}^n) = \oplus_{l=0}^n \wedge^l(\mathbb{R}^n)$ be a graded algebra with respect to the exterior products and $L^p(\Omega, \wedge^l)$ be the set of differential l-forms $u(x) = \sum_I u_I(x) dx_I$ in Ω satisfying $\int_\Omega |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \dots, n$. We denote by $C^\infty(\Omega, \wedge)$ the space of differential forms with coefficients in $C^\infty(\Omega)$. A remarkable advance on operators and differential forms has been made during the recent years, see [2,3,7,23,10,14,19]. Let $|E|$ be the n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$. We use $E \subset\subset F$ to denote that E is strictly contained in F and has the compact closure $\overline{E}$ in F. For a function u, the average of u over B is defined by $u_B = \frac{1}{|B|}\int_B u dx$. We know that Green's operator G is defined on the space of smooth l-forms in Ω by assigning $G(u)$ to be a solution of the Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator, see [1,18,22] for more results about Green's operator. For any subset $E \subset \mathbb{R}^n$ and $p > 1$, we use $W^{1,p}(E, \wedge^l)$ to denote the Sobolev space of l-forms which equals $L^p(E, \wedge^l) \cap L_1^p(E, \wedge^l)$ with norm

$$\|u\|_{W^{1,p}(E)} = \|u\|_{W^{1,p}(E,\wedge^l)} = diam(E)^{-1}\|u\|_{p,E} + \|\nabla u\|_{p,E}. \tag{1.1}$$

The nonlinear partial differential equation for differential forms

$$d^\star A(x, du) = 0 \tag{1.2}$$

is called the (homogeneous) A-harmonic equation, where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ satisfies the conditions:

$$|A(x,\xi)| \le a|\xi|^{p-1} \quad \text{and} \quad A(x,\xi)\cdot\xi \ge |\xi|^p \tag{1.3}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.2). See [1,21,7,23,10,14,19] for more results about the A-harmonic equation.

2. Local higher integrability

In this section, we show the local higher integrability of the compositions of the iterated operators $D^k G^k(u)$ and $D^{k+1} G^k(u)$, where $k = 1, 2, \dots$. These local results will be used to prove the imbedding theorems and the global integrability of these operators in later sections. Let φ be a strictly increasing convex function on $[0, \infty)$, $\varphi(0) = 0$, and u be a differential form defined in a measurable set $E \subset \mathbb{R}^n$ such that $\varphi(\lambda(|u| + |u_E|)) \in L^1(E; \mu)$ for any real number $\lambda > 0$ and $\mu(\{x \in E : |u - u_E| > 0\}) > 0$, where μ is a