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Global weak solutions for the two-dimensional magnetohydrodynamic equations with partial dissipation and diffusion

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ABSTRACT

In this paper, we consider the two-dimensional magnetohydrodynamic equations. We establish global weak solution for MHD equations with partial dissipation and vertical diffusion. We also obtain a regularity condition.

1. Introduction

The viscous incompressible magnetohydrodynamic equations can be written as

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p + \nu_1\,\partial_{xx}u + \nu_2\,\partial_{yy}u + b\cdot\nabla b, \\ b_t + u\cdot\nabla b = \eta_1\,\partial_{xx}b + \eta_2\,\partial_{yy}b + b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ u(x,y,0) = u_0(x,y),\ b(x,y,0) = b_0(x,y) \end{cases} \tag{1.1}$$

where $(x, y) \in \mathbb{R}^2$, $t \geq 0$, $u = (u_1(x, y, t),\ u_2(x, y, t))$ denotes the 2D velocity field, $p = p(x, y, t)$ the pressure, $b = (b_1(x, y, t), b_2(x, y, t))$ the magnetic field, and ν_1, ν_2, η_1 and η_2 are nonnegative real parameters. When $\nu_1 = \nu_2$ and $\eta_1 = \eta_2$, (1.1) reduces to the standard incompressible MHD equations.

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When all four parameters ν_1, ν_2, η_1 and η_2 are positive, the global regularity of 2D MHD equations is well known [13]. However, it remains a remarkable open problem whether classical solutions of the two-dimensional inviscid MHD equation, all four parameters are zero, preserve their regularity for all time or finite time blowup. When $\nu_1 > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 > 0$ or when $\nu_1 = 0$, $\nu_2 > 0$, $\eta_1 > 0$ and $\eta_2 = 0$, the global regularity was established by Cao and Wu [2]. Furthermore, the MHD equations with only magnetic diffusion is shown to possess global weak solutions [2]. Cao, Regmi, and Wu studied two dimensional MHD equations with horizontal dissipation and horizontal diffusion in [1]. They proved that any possible blow-up can be controlled by the L^∞ norm of the horizontal components. Furthermore, they showed that $\|v\|_{L^r}$ with $2 < r < \infty$ at any time does not grow faster than $\sqrt{r \log r}$ as r increases i.e. $\|(u_1, b_1)\|_{L^r} \le C\sqrt{r \log r}$. The MHD equations have been a center of attention to numerous analytical, experimental, and numerical investigations (see [1,2,4–20], and references therein).

In this paper, we consider the MHD equations with partial dissipation in the first component of velocity field and vertical magnetic diffusion.

$$\begin{cases} \partial_t u_1 + u\cdot\nabla u_1 = -\partial_x p + \nu_2 \partial_{yy} u_1 + b\cdot\nabla b_1, \\ \partial_t u_2 + u\cdot\nabla u_2 = -\partial_y p + b\cdot\nabla b_2, \\ \partial_t b + u\cdot\nabla b = \eta_1 \partial_{xx} b + b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ u(x,y,0) = u_0(x,y), \qquad b(x,y,0) = b_0(x,y) \end{cases} \tag{1.2}$$

where ν_2 and η_1 are positive parameters, we assume $\nu_2 = \eta_1 = 1$ for simplicity.

The vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$ satisfy

$$\begin{cases} \omega_t + u\cdot\nabla\omega = -\partial_{yyy} u_1 + b\cdot\nabla j, \\ j_t + u\cdot\nabla j = \partial_{xx} j + b\cdot\nabla\omega + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1). \end{cases} \tag{1.3}$$

We prove the following theorem.

Theorem 1.1. *Assume that $(u_0, b_0) \in H^1(\mathbb{R}^2)$, $\nabla\cdot u_0 = 0$ and $\nabla\cdot b_0 = 0$. Then, (1.2) has a global weak solution (u, b), which satisfies*

$$u, b \in L^\infty([0,T); H^1(\mathbb{R}^2)), \qquad \partial_{yy} u \in L^2([0,T]; L^2(\mathbb{R}^2)), \qquad \partial_x j \in L^2([0,T]; L^2(\mathbb{R}^2))$$

for any $T > 0$.

It is not known if such weak solutions are globally regular. However, if the u_2 satisfies

$$\int_0^T \|\partial_{xx} u_2\|_4 < \infty$$

then the solution becomes a classical solution.

Remark 1.2. • This work is motivated by the recent work in [4,2,1].
• Similar results can be obtained, if we replace $\partial_{yy} u_1$ by $\partial_{xx} u_2$ and $\partial_{xx} b$ by $\partial_{yy} b$ in (1.2).

This paper is organized as follows: we first state some preliminaries and some important lemmas in Section 2 and thereafter in Sections 3 and 4 we prove our main results.

2. Preliminary

Throughout this paper, we use the following notations.

$$\|f\|_{L^2} = \|f\|_2, \qquad \frac{\partial}{\partial x} f = \partial_x f = f_{xx}, \qquad \frac{\partial^2}{\partial x^2} f = \partial_{xx} f, \qquad \int_{\mathbb{R}^2} f\, dxdy = \int f.$$

We state some lemmas which play an important role.