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One-dimensional model equations for hyperbolic fluid flow

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ABSTRACT

In this paper we study the singularity formation for two nonlocal 1D active scalar equations, focusing on the hyperbolic flow scenario. Those 1D equations can be regarded as simplified models of some 2D fluid equations.

1. Introduction

The following transport equation

$$\omega_t + u \cdot \nabla\omega = 0 \tag{1}$$

is a basic mathematical model in fluid dynamics. If u depends on ω, (1) is called an active scalar equation. The problem of deciding whether blowup can occur for smooth initial data becomes very hard if the dependence of ω is nonlocal in space.

The relationship expressing u in terms of ω is commonly called Biot–Savart law. We have the following examples in 2D:

$$u = \nabla^\perp(-\triangle)^{-1}\omega, \tag{2}$$

where $\nabla^\perp = (-\partial_y, \partial_x)$ is the perpendicular gradient. Eqs. (1) and (2) are the vorticity form of 2D Euler equation. When we take

$$u = \nabla^\perp(-\Delta)^{-\frac{1}{2}}\omega,$$

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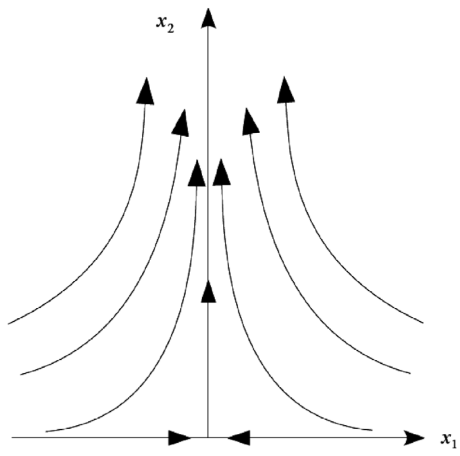

Fig. 1. Illustration of hyperbolic flow scenario in two dimensions.

(1) becomes the surface quasi-geostrophic (SQG) equation, which has important applications in geophysics, or can be regarded as a toy model for the 3D-Euler equations. For more details we refer to [4].

A question of great importance is whether solutions for these equations form singularities in finite time. A promising new approach for the construction of singular solutions is to use the *hyperbolic flow scenario*. In [7,8], such a scenario was proposed to obtain singular solutions for the 3D Euler equations, and in [10], the long-standing question of existence of solutions to the 2D Euler with double-exponential gradient growth was settled using hyperbolic flow.

The hyperbolic flow scenario in two dimensions can be explained in the following way. Consider e.g. a flow in the upper half-plane $\{x_2 > 0\}$. The essential properties required are (see Fig. 1 for an illustration):

- There is a stagnant point of the flow at one boundary point (e.g. the origin) for all times.
- Along the boundary, the flow is essentially directed towards that point for all times.

Such flows can be created by imposing symmetry and other conditions on the initial data. For incompressible flows the stagnant point is a hyperbolic point of the velocity field, hence the name.

The scenario is a natural candidate for creating flows with strong gradient growth or finite-time blowup, since the fluid is compressed along the boundary. Due to non-linear and non-local interactions however, the flow remains hard to control, so a rigorous proof of blowup for the 3D Euler equations using hyperbolic flow remains a challenge. The crucial issue is to stabilize the scenario up to the singular time.

One way to make progress in understanding and to gain insight into the hyperbolic blowup scenario is to study it in the context of one-dimensional model equations. This was begun in [2,1], where one-dimensional models for the 2D-Boussinesq and 3D axisymmetric Euler equations were introduced and blowup was proven.

One-dimensional models capturing other aspects of fluid dynamical equations have a long-standing tradition, one of the earliest being the celebrated Constantin–Lax–Majda model [3]. We refer to the introduction of [11] for a more thorough review of known one-dimensional model equations, and to [1] for discussion of the aspects relating to the hyperbolic flow scenario.

In this paper, we will study 1D models of (1) on $\mathbb{R}$ with the following two choices of u:

$$u_x = H\omega, \tag{3}$$

$$u = (-\Delta)^{-\frac{\alpha}{2}}\omega = -c_\alpha \int_{\mathbb{R}} |y-x|^{-(1-\alpha)}\omega(y,t)\, dy. \tag{4}$$