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# A note on an inequality of Bourgain and Brezis

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HIGHLIGHTS

- A new and elementary proof to an inequality of Bourgain and Brezis is given.
- Avoid using the Littlewood–Paley decomposition.
- An equivalent version of the inequality is given.

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ABSTRACT

In this note, we give a new and elementary approach to an inequality of Bourgain and Brezis (2007) for $L^1$ vector fields, avoid using the Littlewood–Paley decomposition that Bourgain and Brezis used. And we also give an equivalent version of the inequality.

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## 1. Introduction

Inspired by pioneering work [1], many interesting results that involve $L^1$-data have been established. Bourgain and Brezis [2,3] proved the following theorem. Their proof is quite involved, relying on a Littlewood–Paley decomposition.

**Theorem 1.** *Let $n \geq 2$. If $\mathbf{f} \in L^1(\mathbb{R}^n;\mathbb{R}^n), \operatorname{div}\mathbf{f} = 0$ and $\mathbf{u} \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, then there exists a constant $C > 0$, independent of $\mathbf{f}$ and $\mathbf{u}$, such that*

$$\left| \int_{\mathbb{R}^n} \mathbf{f} \cdot \mathbf{u} \mathrm{d}x \right| \leq C \|\mathbf{f}\|_{L^1} \|\nabla \mathbf{u}\|_{L^n}. \tag{1.1}$$

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This theorem has many important applications, see [2,3,11,12,7] for example. Van Schaftingen [10] gave a direct and elementary proof of the above theorem, which uses only the Morrey–Sobolev embedding, and got a slightly more general version. Actually, the proof supplies a slightly stronger result where $\|\nabla \mathbf{u}\|_{L^n}$ can be replaced by $\sum_{i\neq j}\|\frac{\partial u_i}{\partial x_j}\|_{L^n}$, see [10, Remark 3].

**Theorem 2.** *Let $n \geq 2$. If $\mathbf{f} \in L^1(\mathbb{R}^n;\mathbb{R}^n), \operatorname{div}\mathbf{f} \in L^1(\mathbb{R}^n)$ and $\mathbf{u} \in (W^{1,n}\cap L^\infty)(\mathbb{R}^n;\mathbb{R}^n)$, then there exists a constant $C > 0$, independent of $\mathbf{f}$ and $\mathbf{u}$, such that*

$$\left|\int_{\mathbb{R}^n} \mathbf{f}\cdot \mathbf{u}\mathrm{d}x\right| \leq C\left(\|\mathbf{f}\|_{L^1}\sum_{i\neq j}\left\|\frac{\partial u_i}{\partial x_j}\right\|_{L^n} + \|\operatorname{div}\mathbf{f}\|_{L^1}\|\mathbf{u}\|_{L^n}\right).$$

For divergence-free $L^1$ vector fields, the above theorem becomes

**Theorem 3.** *Let $n \geq 2$. If $\mathbf{f} \in L^1(\mathbb{R}^n;\mathbb{R}^n), \operatorname{div}\mathbf{f} = 0$ and $\mathbf{u} \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, then there exists a constant $C > 0$, independent of $\mathbf{f}$ and $\mathbf{u}$, such that*

$$\left|\int_{\mathbb{R}^n} \mathbf{f}\cdot \mathbf{u}\mathrm{d}x\right| \leq C\|\mathbf{f}\|_{L^1}\sum_{i\neq j}\left\|\frac{\partial u_i}{\partial x_j}\right\|_{L^n}.$$

Bourgain and Brezis [3] obtained another slightly sharper version of Theorem 1, which is also stronger than Theorem 3.

**Theorem 4.** *Let $n \geq 2$. If $\mathbf{f} \in L^1(\mathbb{R}^n;\mathbb{R}^n), \operatorname{div}\mathbf{f} = 0$ and $\mathbf{u} \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, then there exists a constant $C > 0$, independent of $\mathbf{f}$ and $\mathbf{u}$, such that*

$$\left|\int_{\mathbb{R}^n} \mathbf{f}\cdot \mathbf{u}\mathrm{d}x\right| \leq C\|\mathbf{f}\|_{L^1}\|\mathrm{CURL}\,\mathbf{u}\|_{L^n}.$$

The original proof of the above theorem in [3] also relies on the Littlewood–Paley decomposition. In fact, with the help of the Gaffney-type inequality and Theorem 1, we find a new approach to prove Theorem 4. Since the elementary proof of Theorem 1 given by Van Schaftingen does not need the Littlewood–Paley decomposition, our method can also avoid using the Littlewood–Paley decomposition. Our new proof will be given in Section 2.

Now we state an equivalent version of Theorem 4.

**Theorem 5.** *Let $n \geq 2$. If $\mathbf{f} \in L^1(\mathbb{R}^n;\mathbb{R}^n), \nabla(-\Delta)^{-1}\operatorname{div}\mathbf{f} \in L^1(\mathbb{R}^n;\mathbb{R}^n)$ and $\mathbf{u} \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, then there exists a constant $C > 0$, independent of $\mathbf{f}$ and $\mathbf{u}$, such that*

$$\left|\int_{\mathbb{R}^n} \mathbf{f}\cdot \mathbf{u}\mathrm{d}x\right| \leq C(\|\mathbf{f}\|_{L^1}\|\mathrm{CURL}\,\mathbf{u}\|_{L^n} + \|\nabla(-\Delta)^{-1}\operatorname{div}\mathbf{f}\|_{L^1}\|\nabla\mathbf{u}\|_{L^n}). \tag{1.2}$$

By a similar way, we can also get a result equivalent to Theorem 1.

**Theorem 6.** *Let $n \geq 2$. If $\mathbf{f} \in L^1(\mathbb{R}^n;\mathbb{R}^n), \nabla(-\Delta)^{-1}\operatorname{div}\mathbf{f} \in L^1(\mathbb{R}^n;\mathbb{R}^n)$ and $\mathbf{u} \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, then there exists a constant $C > 0$, independent of $\mathbf{f}$ and $\mathbf{u}$, such that*

$$\left|\int_{\mathbb{R}^n} \mathbf{f}\cdot \mathbf{u}\mathrm{d}x\right| \leq C(\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1}\operatorname{div}\mathbf{f}\|_{L^1})\|\nabla\mathbf{u}\|_{L^n}. \tag{1.3}$$

**Remark 7.** By Theorem 6 and standard elliptic estimates, we can see that [6, Proposition 1] still holds for $q = n'$, where $n' = \frac{n}{n-1}$.

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