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Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity

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ABSTRACT

This paper deals with the existence and the asymptotic behavior of non-negative solutions for a class of stationary Kirchhoff problems driven by a fractional integro-differential operator $\mathcal{L}_K$ and involving a critical nonlinearity. In particular, we consider the problem

$$-M(\|u\|^2)\mathcal{L}_K u = \lambda f(x,u) + |u|^{2_s^*-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, 2_s^* is the critical exponent of the fractional Sobolev space $H^s(\mathbb{R}^n)$, the function f is a subcritical term and λ is a positive parameter. The main feature, as well as the main difficulty, of the analysis is the fact that the Kirchhoff function M could be zero at zero, that is the problem is *degenerate*. The adopted techniques are variational and the main theorems extend in several directions previous results recently appeared in the literature.

1. Introduction

In the last years, the interest towards nonlinear boundary value stationary Kirchhoff problems has grown more and more, thanks in particular to their intriguing analytical structure due to the presence of the nonlocal Kirchhoff function M which makes the equation no longer a pointwise identity. In the present paper we consider the problem

$$-M\left(\|u\|^2\right)\mathcal{L}_K u = \lambda f(x,u) + |u|^{2_s^*-2}u \quad \text{in } \Omega,$$

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$$u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \tag{1.1}$$
$$\|u\|^2 = \iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)dxdy,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n > 2s$, with $s \in (0,1)$, the number $2_s^* = 2n/(n-2s)$ is the critical exponent of the fractional Sobolev space $H^s(\mathbb{R}^n)$, the function f is a subcritical term and λ is a positive parameter.

The main nonlocal fractional operator $\mathcal{L}_K$ is defined for any $x \in \mathbb{R}^n$ by

$$\mathcal{L}_K\varphi(x) = \frac{1}{2}\int_{\mathbb{R}^n} (\varphi(x+y) + \varphi(x-y) - 2\varphi(x))K(y)dy, \tag{1.2}$$

along any $\varphi \in C_0^\infty(\Omega)$ and the kernel $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ is a *measurable* function for which

(K_1) $m\,K \in L^1(\mathbb{R}^n)$ *with* $m(x) = \min\left\{|x|^2, 1\right\}$;
(K_2) *there exists* $\theta > 0$ *such that* $K(x) \ge \theta\,|x|^{-(n+2s)}$ *for any* $x \in \mathbb{R}^n \setminus \{0\}$;

hold.

A typical example of K is given by $K(x) = |x|^{-(n+2s)}$. In this case the operator $\mathcal{L}_K = -(-\Delta)^s$ reduces to the fractional Laplacian, which (up to normalization factors) may be defined for any $x \in \mathbb{R}^n$ as

$$(-\Delta)^s\varphi(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{2\varphi(x) - \varphi(x+y) - \varphi(x-y)}{|y|^{n+2s}}dy$$

along any $\varphi \in C_0^\infty(\Omega)$; see [13,20,21,32,33] and references therein for further details on the fractional Laplace operator $(-\Delta)^s$ and the fractional Sobolev spaces $H^s(\mathbb{R}^n)$ and $H_0^s(\Omega)$.

Several recent papers are focused both on theoretical aspects and applications related to nonlocal fractional models. Concerning the critical case, in [5] the effects of lower order perturbations are studied for the existence of positive solutions of critical elliptic problems involving the spectral fractional Laplacian and not via formula (1.2). In [6], see also the references therein, existence and multiplicity results are established for a critical fractional equation with concave–convex nonlinearities. In [31] a Brézis–Nirenberg existence result for nonlocal fractional equations is proved through variational methods, while in [30] the authors extend the theorems obtained in [31] to a more general problem, again involving an integro-differential nonlocal operator and critical terms. Furthermore, a multiplicity result for a Brézis–Nirenberg problem in nonlocal fractional setting is given in [17], where it is shown that in a suitable left neighborhood of any eigenvalue of $\mathcal{L}_K$ (with Dirichlet boundary data) the number of solutions is at least twice the multiplicity of the eigenvalue. For multiplicity results on a *non-degenerate* stationary Kirchhoff problem involving $\mathcal{L}_K$ and a nonlinearity of integral form we refer to [23] and the references therein.

For evolutionary Kirchhoff problems it is worth mentioning the paper [2], which is concerned with lifespan estimates of maximal solutions for *degenerate* polyharmonic Kirchhoff problems. The technique goes back to [28] and it is based on the construction of a Lyapunov function $\mathscr{L}$ which lives as long as any local solution u does. The goal is to show that $\mathscr{L}$ becomes unbounded in finite time, proving the non-continuation of u. A priori estimates for the maximal living time T are obtained exploiting in a suitable way the non-existence tools already adopted in [4] for possibly *degenerate* $p(x)$-Kirchhoff systems involving nonlinear damping and source terms. More recently, in [12] a multiplicity result is obtained for a *degenerate* stationary Kirchhoff problem governed by the $p(x)$-polyharmonic operator, with a subcritical term, through the mountain pass theorem, while in [3] existence and multiplicity of solutions of certain eigenvalue stationary p-polyharmonic Kirchhoff problems are considered, also in a *degenerate* setting.

Furthermore, the very interesting paper [27] treats the question of the existence and multiplicity of non-trivial non-negative entire solutions of a Kirchhoff eigenvalue problem, involving critical nonlinearities and