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Boundary theory on the Hata tree

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ABSTRACT

We prove that for a certain Markov chain on the symbolic space of the Hata tree K, the Martin boundary $\mathcal{M}$ is homeomorphic to the *trunk* of the Hata tree, and the minimal Martin boundary is the post-critical set $\{1\dot{2}, \dot{1}, \dot{2}\}$, which corresponds to the three vertices of the trunk. Moreover, the class of P-harmonic functions on $\mathcal{M}$ coincides with Kigami's class of harmonic functions on K.

1. Introduction

Recently there are considerable interests in studying various Markov chains or random walks defined on the symbolic space of a self-similar set K and in identifying K with the Martin boundary $\mathcal{M}$ of the chain [1–11]. In this way one can carry the natural harmonic structure on the Martin boundary onto the self-similar set K. In particular it is conceivable that with a suitable Markov chain, one can obtain an induced harmonic structure and a Laplacian that coincide with the classical ones on the self-similar set constructed by Kigami [6].

In [9], the authors considered the Sierpinski gasket (SG) and introduced a Markov chain on the symbolic space of the SG as follows. Let $\Sigma = \{1, 2, 3\}$, Σ_n be the set of finite words of length n, and Σ_* be the set of all finite words. The Markov chain $\{X_k\}_{k=0}^{\infty}$ on Σ_* is defined in such a way that once X_k reaches one of the three vertices of each Σ_n, it will jump down, with equal probability, to its three descendants in Σ_{n+1}, while if X_k is not at the three vertices, it will move to its three neighbors with equal probability. We call the harmonic functions induced on $\mathcal{M}$ the *P-harmonic functions*, and those in Kigami's construction the *K-harmonic functions*. The main conclusion in [9] is the following.

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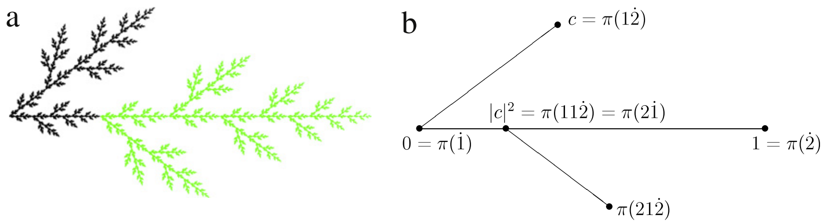

Fig. 1. (a) The Hata tree with $c = 0.4 + 0.3i$. (b) Projections of the critical set $\{11\dot{2}, 2\dot{1}\}$ and the post-critical set $\mathcal{P} = \{1\dot{2}, \dot{1}, \dot{2}\}$. $\pi(\mathcal{P})$ is the boundary $\mathcal{V}_0 = \{c, 0, 1\}$.

Theorem 1.1. *For the above Markov chain, the Martin boundary $\mathcal{M}$ is homeomorphic to the SG, and the minimal boundary $\mathcal{M}_{\min}$ corresponds to the three vertices of the SG. Moreover, under this homeomorphism, the P-harmonic functions coincide with the K-harmonic functions.*

It is a natural question whether the results in Theorem 1.1 can be extended to all post-critically finite self-similar sets. Here we show that an analogous construction for the Hata tree yields a Martin boundary that is not homeomorphic to the entire self-similar set, but to a proper subset of it.

As is well known, the general result for the existence of the Laplacian in Kigami's approach applies to post-critically finite self-similar sets with some very strong symmetry properties. It is an open question whether such symmetry conditions can be removed. In this note we give a detailed study of the Hata tree and the Martin boundary, with a Markov chain similar to the one for the SG. Note that the Hata tree is one of the simplest non-symmetric self-similar sets. We use it to test our Martin boundary approach and gain more insights to the existence of the Laplacian for non-symmetric self-similar sets.

Recall that a *Hata tree* (see [12]) is the attractor K of an *iterated function system (IFS)* on $\mathbb{C}$ consisting of two similitudes

$$S_1(z) = c\bar{z}, \qquad S_2(z) = (1 - |c|^2)\bar{z} + |c|^2, \tag{1.1}$$

where $|c| < 1$, $|1 - c| < 1$ and $\mathrm{Im}(c) \neq 0$ (Fig. 1(a)). According to [12], K is a post-critically finite (p.c.f.) self-similar set. Its boundary $\mathcal{V}_0 = \{c, 0, 1\}$ consists of three points which correspond to the symbols $1\dot{2}$, $\dot{1}$, $\dot{2}$ (Fig. 1(b)). We will assume (1.1) satisfies the open set condition. A sufficient condition for this is that [13]: for $c = re^{i\theta}$ and $0 < \theta < \pi$,

$$\cos\theta > -r^3/(2(1 - r^2)).$$

We let $T_s := [0, c]$ and $T_h := [0, 1]$ denote the slant and horizontal line segments in K (see Fig. 1(a)) and call $T = T_s \cup T_h$ the *trunk* of the tree. A *branch* of K is defined as a connected component of $K \setminus T$, and a *node* is the intersection of the trunk and a branch. Let $\{X_k\}_{k=0}^{\infty}$ be the Markov chain analogous to the one on the SG in [9] (see Section 2 for the exact definition). We have

Theorem 1.2. *Let $\mathcal{M}$ be the Martin boundary of $\{X_k\}_{k=0}^{\infty}$. Then there is a canonical continuous map $\iota : K \to \mathcal{M}$ such that ι is a homeomorphism when restricted to T, and $\iota(\mathcal{V}_0) = \mathcal{M}_{\min}$. Moreover for each branch B of K with node b, $\iota(B) = \{\iota(b)\}$.*

It is seen that the Hata tree differs significantly from the SG in that the Martin boundary corresponds to the trunk, and each branch is identified as a point. Nevertheless the harmonic functions are as expected.

Proposition 1.3. *Under the above map ι, the P-harmonic functions coincide with the K-harmonic functions. More precisely, a K-harmonic function is constant on each branch, and the induced map ι_* defined by*

$$\iota_*(f)(x) = f(\iota(x))$$

is an isomorphism from the class of P-harmonic functions to the class of K-harmonic functions.

The main approach of proof is similar to that in [9], and the unexplained notation and ideas in the sequel can be found there. The major difference is that for any two points on a branch, the chain has to go through the same node in order to reach a boundary vertex (see definition in Section 2), and the Martin metric on the Martin boundary cannot distinguish such points. On the other hand this allows us to simplify much of the proof (Lemma 2.1) as the random walk to the vertices can be reduced to the gambler's ruin problem.

2. Preliminaries

Let K be the Hata tree generated by the IFS in (1.1). We define some standard notation associated with the IFS. For $n \geq 0$, let $\Sigma_n := \{1, 2\}^n$ with $\Sigma_0 := \{\vartheta\}$. Also, let $\Sigma_* := \bigcup_{n=0}^{\infty} \Sigma_n$ and $\Sigma_\infty := \{1, 2\}^\infty$. For $x = i_1 \cdots i_n \in \Sigma_n$, we let $|x| = n$ denote the *length* of x, and for $k \leq n$, let $x|_k = i_1 \cdots i_k$ denote the initial segment (or *prefix*) of x of length k. For $\mathbf{x} = i_1 i_2 \cdots \in \Sigma_\infty$, let $\mathbf{x}|_n := i_1 \cdots i_n$ be defined similarly, and let $K_{\mathbf{x}|_n} = S_{\mathbf{x}|_n}(K) := S_{i_1} \circ \cdots \circ S_{i_n}(K)$. Define the *projection* $\pi : \Sigma_\infty \to K$ by $\{\pi(\mathbf{x})\} = \bigcap_{n=0}^{\infty} K_{\mathbf{x}|_n}$. It follows that for any $x_0 \in \mathbb{C}$, $\pi(\mathbf{x}) = \lim_{n\to\infty} S_{i_1} \circ \cdots \circ S_{i_n}(x_0)$.