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BLOWUP OF SOLUTIONS FOR A CLASS OF QUASILINEAR EVOLUTION EQUATIONS*

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1. INTRODUCTION

The equation

$$u_{tt} - u_{xxt} = \sigma(u_x)_x, \; x \in (0, 1), t > 0 \tag{1}$$

is used to model the longitudinal displacement in a homogeneous bar, which is composed of a viscoelastic material of the rate type and is of uniform cross section and unit length. Where the term u_{xxt}, arising from the viscoelastic material of the bar, makes the initial boundary value problem of equation (1) more tractable than that of the equation of one-dimensional nonlinear elasticity

$$u_{tt} = \sigma(u_x)_x. \tag{2}$$

It is well-known that even for smooth initial data, global smooth solutions to (2) do not, in general, exist [1], while the question of the existence and uniqueness of solutions to (1) augmented with various types of boundary conditions and initial conditions has been dealt with in many papers, see [2–6]. These papers all establish the existence and uniqueness of global solutions to (1), some get the classical solution, while others obtain the generalized solution. In each paper, $\sigma(s)$ is assumed to be smooth and monotonic, i.e. $\sigma'(s) > 0$ for all $s \in R$, and the initial data are smooth. More recently, (1) has been considered under a somewhat relaxed restriction, i.e. $\sigma'(s) \geq C$ for all $s \in R$, where C is a fixed constant, and the global generalized solutions to (1) have been obtained, see [7,8]. But, if $\sigma(s)$ is not monotonically increasing or $\sigma'(s)$ is not bounded below, for instance, $\sigma(s) = as^p$, a simple type of function, where $a(\neq 0)$ and $p(> 1)$ are all real numbers, except $a > 0$ and $p = 2m + 1 (m = 0, 1, \ldots)$, neither is $\sigma'(s)$ positive nor $\sigma'(s)$ is bounded below, so all the above-mentioned results in paper [2–8] cease to be effective, an open problem is whether the global solutions of the initial boundary value problems of (1) still exist in this circumstance.

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In this paper, we consider (1) together with the initial conditions

$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), x \in (0,1), \tag{3}$$

and one of the following boundary conditions

$$u(0,t) = u(1,t) = 0, t > 0, \tag{4}$$

$$u(0,t) = u_x(1,t) = 0, t > 0, \tag{5}$$

$$u_x(0,t) = u(1,t) = 0, t > 0, \tag{6}$$

and

$$u_x(0,t) = u_x(1,t) = 0, t > 0. \tag{7}$$

For the above-mentioned initial boundary value problems (which we call IBVP), first, assume that $\sigma(s) = a|s|^{p-1}s$ in (1), by an energy estimate method, we prove that the corresponding IBVP, except the Neumann boundary condition (7), have no global solution u so long as $a < 0, 1 < p \le 5$ and the initial functions satisfy certain conditions, i.e. there is a finite constant $\tilde{T}$ such that $\|u\|^2_{L_2[0,1]} + \int_0^t \|u_x\|^2_{L_2[0,1]} dt \to +\infty$ as $t \to \tilde{T}^-$. Next, assume that $\sigma(s) = as^p$ in (1), where $a \neq 0, p$ is an even number, with the help of Jensen's inequality we also prove that the above-mentioned IBVP, except Dirichlet boundary value condition (4), have no global solutions under certain assumptions on initial data. Last, for the first boundary value problem (1), (3), (4), assume that $\sigma(s) = as^p$, by Fourier transform method and taking advantage of the topological invariance of some spaces which we will establish, we prove that the solution u of the problem (1), (3), (4) blows up in finite time $\tilde{T}$, i.e. $u_x(0,t) \to \infty$ and $\|u_x\|_{L_2[0,1]} \to \infty$ as $t \to \tilde{T}^-$ so long as $a \neq 0, p > 2, p$ is an integer (except $a > 0$ and $p = 2m+1, m = 0, 1, \dots$), and the initial data satisfy some conditions.

This paper is divided into five sections. A full statement of the results, along with some definitions and remarks on notations, is given in Section 2. The proofs of the blowup theorems based respectively on an energy estimate method and Jensen's inequality are given in Section 3. And making use of the Fourier transform method, we prove another blowup theorem for the first boundary value problem (1), (3), (4) in Section 5 after some lemmas in Section 4.

2. THE STATEMENT OF MAIN RESULTS

We denote the norm of the space $L_2[0,1]$ by $\|\cdot\|$ and define the Fourier transform $\wedge : L_2[0,1] \to l^2, \forall f(x) \in L_2[0,1], \hat{f}(k) = 2\int_0^1 f(x)\cos k\pi x \, dx = f_k, k = 0, 1, \dots$. Let $\tilde{f} = (f_0, \dots, f_k, \dots)$, then $\tilde{f} \in l^2$. Obviously, according to our definition, $f_{-k} = f_k, f(x) = f_0/2 + \sum_{k=1}^\infty f_k \cos k\pi x$, and $2\|f\|^2 = f_0^2/2 + \sum_{k=1}^\infty f_k^2$. We provide that the notation $\tilde{f} \ge 0$ means $f_k \ge 0, k = 0, 1, \dots$, and a similar notation is used by $\tilde{f} \le 0$.

Let $D = \{v(x) \mid v(x) \in H^2[0,1], v'(0) = v'(1) = 0\}$, then D is a Hilbert space under the norm $\|v\|_D = \|v\|_{H^2[0,1]} = (\|v\|^2 + \|v_{xx}\|^2)^{1/2}$, the sequence $\{e_0 = 1/2, e_k = \cos k\pi x\}_{k=1}^\infty$ is an orthogonal basis in $L_2[0,1]$ and at the same time in D, for $\forall\ v(x) \in D, v(x) = \sum_{k=0}^\infty v_k e_k$, where $v_k = \hat{v}(k)$. Obviously, the corresponding $\tilde{v} = (v_0, v_1, \dots, v_k \dots)$. Let $\tilde{D} = \{\tilde{v} \mid \tilde{v} \in l^2, (0, v_1, \dots, k^2 v_k, \dots) \in l^2\}$ and $\tilde{D}$ be equipped with the norm

$$\|\tilde{v}\|_{\tilde{D}} = \|v\|_D = \left[\frac{1}{2}\left(v_0^2/2 + \sum_{k=1}^\infty (1+k^4)v_k^2\right)\right]^{1/2},$$