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# Crossing-effect in non-isolated and non-symmetric systems of patches

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ABSTRACT

The main result of this article is the crossing-effect between two fragments with no isolation on its sides. The crossing-effect term is given to the phenomenon of relative inversion between the minimum sizes of two patches, only varying the growth rate equally in both. This phenomenon was found from the determination of the minimal size prediction for the general case of problems with two identical patches. This prediction is presented in the explicit form, which allows to recuperate all the cases found in the literature as particular cases, namely, one isolated fragment, one single fragment communicating with its neighborhood, a system with two identical fragments isolated from the matrix but mutually communicating and a system of two identical fragments inserted in a homogeneous matrix. To find the crossing-effect, a particular case of the general solution is approached, which is a single fragment communicating with the matrix with different life difficulties on each side. To verify this statement, it is proposed an experiment, which is an adaptation of other experiments in the literature. The confirmation of the phenomenon presented in this work would be new and unexpected, on the other hand, the refutation of this phenomenon would bring worries to the minimum size models using FKPP equation.

## 1. Introduction

Populations living in one (Nelson and Shnerb, 1998) or more (Lopez and Bonasora, 2017) fragments have been the subject of study in many natural sciences, in particular Ecology (Ferraz et al., 2007). In fact, natural scientists such as physicists (Artiles et al., 2008; Kumar and Kenkre, 2011), mathematicians (Cantrell et al., 1991; Hening et al., 2017) and others (Skellam, 1951) have been studied models that can describe, with some degree of approximation, the real phenomenon.

The study of fragmented regions problem was started by Skellam (1951), which proposed a solitary fragment that can harbor life within the patch while outside it, life is impossible. This problem was improved by Ludwig et al. (1979). They introduced a non-hard region outside the patch in the Skellam problem. Then, in the Ludwig problem, the population in study can go outside the island, but cannot live there forever. The improvement of Ludwig et al. brought the need of a smaller fragment than the one found by Skellam to enable stable life in the system with only one fragment. The natural sequence to Ludwig work is the introduction of another fragment in the system. Now, there are two fragments communicating by a region not favorable to life, but not infinitely hard, in such a way that is possible population elements pass from one fragment to another. In this sense, there are previous studies for the implicit form to minimal size fragments that enables life for a more general case (Pamplona da Silva and Kraenkel, 2012). The explicit form for this minimal size was presented by Kenkre and Kumar (2008) to the case where the life difficulty was the same outside the system and between the patches.

In this article is proposed one general analytic prediction to minimal size of each fragment in a system of two identical patches. This result enables to confirm a previous result (Pamplona da Silva and Kraenkel, 2012), here called "crossing-effect". The crossing-effect term is related to the intersection of two curves in the graphic of fragment size versus growth rate. The crossing of the curves may seem only a mathematical curiosity, but it is more than it.

In order to better explain the crossing-effect phenomenon associated to this crossing of the two curves, it is used two fictitious fragments, $W$ and $Z$, with very specific and different characteristics for the same species and the same growth rate $a_0$. For small $a_0$ $W$ can provide the stable life inside it with a smaller size than $Z$. This fact is due to other parameters of the fragments, as it is well established in the literature (Kenkre and Kumar, 2008; Pamplona da Silva and Kraenkel, 2012) and is even intuitive. Knowing that $W$, for small $a_0$, should be smaller than $Z$, the novelty here presented is that if the growth rate is increased, maintaining the other characteristics of the fragments, $W$ should be greater than $Z$ to provide life within themselves. Briefly, the increase of the growth rate within the fragments produces a relative inversion in the minimum sizes of the fragments.

The predictions proposed above were found in one dimension, where the focus was the minimum length for life existence. In two dimensions, the study would concern the minimal area (Azevedo and

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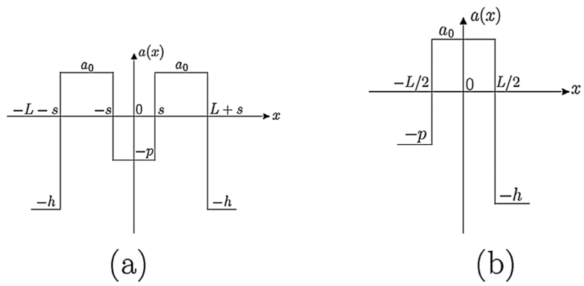


**Fig. 1.** Representation of two identical patches with length $L$ and internal condition $a_0$ immersed in a matrix with life difficulty $h$, separated by a region with life difficulty $p$ and (a) of length $s$ that (b) approaches a non-symmetric single patch when $s \to \infty$.

Kraenkel, 2012) and it would be necessary to explore the geometry of the fragment (Kenkre and Kumar, 2008), which would result in another scope of work, not addressed in this paper. All cases mentioned previously use (Fisher, 1937) Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) equation to describe the behavior of a population density $u(x, t)$, in the time ($t$), moving in the space ($x$), governed by the growth rate $a$ and contained by the saturation rate $b$. In one-dimensional it is given by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + a(x)u - \mathrm{b}u^2, \tag{1}$$

where $D$ is the diffusion coefficient. The parameter $a$ is the growth rate and can be used to describe the environmental conditions of a region. The parameter $b$ represents a population intraspecific competition which defines the maximum population that can occupy a specific fragment. Parameter $b$ is closely linked to the carrying capacity.

## 2. Mathematical predictions

The mathematical purpose of this article is, starting from Eq. (1), to find the general case to minimal size of each patch in a system of two identical patches – see Fig. 1a – and recover several cases addressed in the literature. In addition, it is found the minimal size to single non-symmetric patch able to sustaining life.

Eq. (1) does not have general solution for an arbitrary function $a(x)$ at any time $t$, but there are two considerations that can help to reach the goal without finding the general solution. The first one considers the stationary situation of Eq. (1), because this represents the situation after the transient solution, when there is a stable solution (population) or the solution is null (extinct population). The second consideration is to suppress the nonlinear term $(-u^2)$. One justification for this consideration is that, if population extinction occurs without considering the saturation, the population will die more quickly when this factor is considered. In addition, intraspecific competition is important when the population is large, however it is insignificant when the population is small. Thus, once when the patch size is close to its minimum size, a small stable population is expected and the saturation term may be disregarded. These two considerations are in accordance with the Ludwig ideas (Ludwig et al., 1979), reinforced by Kenkre and Kumar (2008) and previous works (Pamplona da Silva and Kraenkel, 2012; Pamplona da Silva et al., 2017) resulting an ordinary differential equation, that can be written as follow:

$$D\frac{d^2\Phi}{dx^2} + a(x)\Phi = 0. \tag{2}$$

By choosing the function $a(x)$ as a piecewise constant, Eq. (2) can be solved easily in any constant piece. It describes a heterogeneous region (Kenkre and Kumar, 2008; Kenkre and Kuperman, 2003; Kraenkel and Pamplona da Silva, 2010; Pamplona da Silva and Kraenkel, 2012), where $a(x) > 0$ represents a patch (life region) and $a(x) < 0$ represents a dead region. Note that there are space heterogeneities, but the condition inside each patch is homogeneous and the outside conditions are also homogeneous. Then, there are only abrupt changes in the environment condition and they occur in the frontiers between a patch and a death region. The generalization of two communicating identical patches is represented by Fig. 1a.

In order to reach the main mathematical objective of this work, which is to find the explicit form of the minimum size to each fragment represented in Fig. 1a that allows the existence of stable life inside the patch, it is solved the linear equation (Eq. (2)) in each region where $a(x)$ is constant and it is imposed that the function and its first derivative are continuous at the boundaries between fragments and inhospitable regions, following the ideas of Ludwig et al. and their successors (Kenkre and Kumar, 2008; Ludwig et al., 1979; Pamplona da Silva et al., 2017).

These conditions are applied only for the positive values of $x$ ($x = s$ and $x = L + s$) because, from symmetry, the negative side returns the same conditions. In order to simplify the notation, it is introduced

$$\alpha_a = \sqrt{\frac{a}{D}}, \quad \forall\, a. \tag{3}$$

Solutions to Eq. (2) are:

$$\Phi_I(x) = A\cosh(\alpha_p x), \quad \text{in region I } (-s < x < s); \tag{4}$$

$$\Phi_{\mathrm{II}}(x) = B\cos\left(\alpha_{a_0}x + \phi\right), \quad \text{in region II } (s < x < L + s); \tag{5}$$

$$\Phi_{\mathrm{III}}(x) = \mathrm{C}e^{-\alpha_h x}, \quad \text{in region III } (x > L + s). \tag{6}$$

Next, it is imposed the matching condition upon $\Phi(x)$ at each boundary. In fact:

- At $x = s$, the condition is given by $\Phi_{II}(s) = \Phi_I(s)$, then:

$$B\cos\left(\alpha_{a_0}s + \phi\right) = A\cosh(\alpha_p s). \tag{7}$$

- At $x = L + s$, the condition $\Phi_{II}(L + s) = \Phi_{III}(L + s)$ gives:

$$B\cos\left(\alpha_{a_0}L + \alpha_{a_0}s + \phi\right) = \mathrm{C}e^{-\alpha_h(L+s)}. \tag{8}$$

Not only the function $\Phi(x)$ should be continuous but also its derivative. Hence:

- In $x = s$, immediately:

$$-B\alpha_{a_0}\sin\left(\alpha_{a_0}s + \phi\right) = A\alpha_p\sinh(\alpha_p s). \tag{9}$$

- In $x = L + s$:

$$-B\alpha_{a_0}\sin\left(\alpha_{a_0}L + \alpha_{a_0}s + \phi\right) = -C\alpha_h e^{-\alpha_h(L+s)}. \tag{10}$$

Dividing Eq. (9) by Eq. (7), it follows:

$$-\alpha_{a_0}\tan\left(\alpha_{a_0}s + \phi\right) = \alpha_p\tanh(\alpha_p s) \Rightarrow$$

$$-\alpha_{a_0}s - \phi = \arctan\left[\frac{\alpha_p}{\alpha_{a_0}}\tanh\alpha_p s\right]. \tag{11}$$

Dividing Eq. (10) by Eq. (8), it results:

$$-\alpha_{a_0}\tan\left(\alpha_{a_0}L + \alpha_{a_0}s + \phi\right) = -\alpha_h \Rightarrow$$

$$\alpha_{a_0}L + \alpha_{a_0}s + \phi = \arctan\left(\frac{\alpha_h}{\alpha_{a_0}}\right). \tag{12}$$

Eqs. (11) and (12) can be added. The convenient form of the result from this operation is: