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On the diameter of an ideal

Michela Di Marca, Matteo Varbaro *

Dip. di Matematica, Università di Genova, Genova, Italy

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A B S T R A C T

We begin the study of the notion of diameter of an ideal $I \subset S = \Bbbk[x_1, \dots, x_n]$, an invariant measuring the distance between the minimal primes of I. We provide large classes of Hirsch ideals, i.e. ideals satisfying $\operatorname{diam}(I) \leq \operatorname{height}(I)$, such as: quadratic radical ideals such that S/I is Gorenstein and $\operatorname{height}(I) \leq 4$, or ideals admitting a square-free complete intersection initial ideal.

Introduction

The dual graph is a classical tool introduced in different contexts, as in algebraic geometry or in combinatorics, in order to study intersection patterns of algebraic curves, or combinatorial properties of simplicial complexes. More in general, it is possible to define the concept of dual graph for ideals in a Noetherian commutative ring. Under different names, this natural notion has already been studied by several authors, such as [7,8,4,13].

Let $S = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field $\Bbbk$. For an ideal $I \subset S$, let $\operatorname{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ be the set of minimal primes of I. The *dual graph* $G(I)$ of I is the graph $G = ([s], E)$, where $[s] := \{1, 2, \dots, s\}$ and

* Corresponding author.
E-mail addresses: dimarca@dima.unige.it (M. Di Marca), varbaro@dima.unige.it (M. Varbaro).

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$$\{i, j\} \in E \Leftrightarrow \text{height}(\mathfrak{p}_i + \mathfrak{p}_j) = \text{height}(I) + 1.$$

In this paper, we are particularly interested in studying the diameter of $G(I)$, that we will name the *diameter of* I and denote by $\text{diam}(I)$, in the homogeneous case. In general the diameter of I can be infinite. If $\text{diam}(I) < \infty$, i.e. if $G(I)$ is connected, then I must be height-unmixed, that is $\text{height}(\mathfrak{p}) = \text{height}(I)\ \forall\ \mathfrak{p} \in \text{Min}(I)$. In this case, the number of vertices s is at most the multiplicity $e(S/I)$ of S/I. Since a connected graph has diameter less than the number of vertices, then $\text{diam}(I) < e(S/I)$. Synthetically, so:

$$\text{diam}(I) < \infty \implies \text{diam}(I) < e(S/I).$$

From a result of Hartshorne in [7], it follows that, if $I \subset S$ is homogeneous and S/I is Cohen–Macaulay, then $\text{diam}(I) < \infty$. By the above discussion, therefore, in this case $\text{diam}(I) < e(S/I)$. This upper bound can be significantly improved in good situations. For monomial ideals satisfying the Serre's condition (S_2), for example, much better bounds are described in [9].

In this spirit, we say that an ideal $I \subset S$ is *Hirsch* if $\text{diam}(I) \leq \text{height}(I)$. One cannot expect more: an easy ideal such as $I = (x_1y_1, \ldots, x_ny_n) \subset \Bbbk[x_i, y_i : i = 1, \ldots, n]$, which is a square-free monomial complete intersection, satisfies $\text{diam}(I) = \text{height}(I)$.

The name comes from a conjecture made by Hirsch in 1957. He conjectured that if a simplicial complex Δ is the boundary of a convex polytope, then its Stanley–Reisner ideal I_Δ is Hirsch. The conjecture has been disproved by Santos in [12].

However, under some additional hypotheses, there are some positive answers. For example, Adiprasito and Benedetti proved Hirsch's conjecture when Δ is flag in [1]. More generally, they proved that if I is monomial, quadratic, and S/I satisfies Serre's condition (S_2), then I is Hirsch. Starting from this, in [4] Benedetti and the second author made the following general conjecture:

Conjecture. *Let $I \subset S$ be a homogeneous ideal generated in degree 2. If S/I is Cohen–Macaulay, then I is Hirsch.*

Beyond monomial ideals, the conjecture is known to be true in other situations. For example, recently Bolognini, Macchia and Strazzanti checked it for binomial edge ideals of bipartite graphs in [5]. Motivated by the above conjecture, in this paper we provide large classes of Hirsch ideals:

Main Theorem. *A radical homogeneous ideal $I \subset S$ is Hirsch in the following cases:*

i) *If S/I is Gorenstein and* $\text{reg}\, S/I \leq 3$.
ii) *If S/I is Gorenstein and I is an ideal generated in degree 2 of height ≤ 4.*
iii) *If S/I is Gorenstein and I is an ideal generated in degree 2 of height 5 which is not a complete intersection.*