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Artinian level algebras of socle degree 4 ☆

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ABSTRACT

In this paper we study the O-sequences of local (or graded) K-algebras of socle degree 4. More precisely, we prove that an O-sequence $h = (1, 3, h_2, h_3, h_4)$, where $h_4 \geq 2$, is the h-vector of a local level K-algebra if and only if $h_3 \leq 3h_4$. A characterization is also presented for Gorenstein O-sequences. In each of these cases we give an effective method to construct a local level K-algebra with a given h-vector. Moreover we refine a result of Elias and Rossi by showing that if $h = (1, h_1, h_2, h_3, 1)$ is a unimodal Gorenstein O-sequence, then h forces the corresponding Gorenstein K-algebra to be canonically graded if and only if $h_1 = h_3$ and $h_2 = \binom{h_1+1}{2}$, that is the h-vector is maximal. We discuss analogue problems for higher socle degrees.

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1. Introduction

Let $(A, \mathfrak{m})$ be an Artinian local or graded K-algebra where K is any arbitrary field unless otherwise specified. Let $\operatorname{Soc}(A) = (0 : \mathfrak{m})$ be the socle of A. We denote by s the *socle degree* of A, that is the maximum integer j such that $\mathfrak{m}^j \neq 0$. The *type* of A is $\tau := \dim_K \operatorname{Soc}(A)$. Recall that A is said to be *level* of type τ if $\operatorname{Soc}(A) = \mathfrak{m}^s$ and $\dim_K \mathfrak{m}^s = \tau$. If A has type 1, equivalently $\dim_K \operatorname{Soc}(A) = 1$, then A is *Gorenstein*. In the literature local rings with low socle degree, also called *short local rings*, have emerged as a testing ground for properties of infinite free resolutions (see [1], [2], [10], [20], [32], [35]). They have been also extensively studied in problems related to the irreducibility and the smoothness of the punctual Hilbert scheme $\operatorname{Hilb}_d(\mathbb{P}^n_K)$ parameterizing zero-dimensional subschemes in $\mathbb{P}^n_K$ of degree d, see among others [7], [8], [16], [31]. In this paper we study the structure of level K-algebras of socle degree 4, hence $\mathfrak{m}^5 = 0$. One of the most significant information on the structure is given by the Hilbert function.

By definition, the Hilbert function of A,

$$h_i = h_i(A) := \dim_K \mathfrak{m}^i/\mathfrak{m}^{i+1},$$

is the Hilbert function of the associated graded ring $gr_{\mathfrak{m}}(A) := \oplus_{i\geq 0}\mathfrak{m}^i/\mathfrak{m}^{i+1}$. We also say that $h = (h_0, h_1, \ldots, h_s)$ is the h-vector of A. In [29] Macaulay characterized the possible sequences of positive integers h_i that can occur as the Hilbert function of A. Since then there has been a great interest in commutative algebra in determining the h-vectors that can occur as the Hilbert function of A with additional properties (for example, complete intersection, Gorenstein, level, etc). A sequence of positive integers $h = (h_0, h_1, \ldots, h_s)$ satisfying Macaulay's criterion, that is $h_0 = 1$ and $h_{i+1} \leq h_i^{\langle i\rangle}$ for $i = 1, \ldots, s-1$, is called an O-sequence. A sequence $h = (1, h_1, \ldots, h_s)$ is said to be a *level (resp. Gorenstein) O-sequence* if h is the Hilbert function of some Artinian level (resp. Gorenstein) K-algebra A. Remark that h_1 is the embedding dimension and, if A is level, h_s is the type of A. Notice that a level O-sequence is not necessarily the Hilbert function of an Artinian level graded K-algebra. This is because the Hilbert function of the level ring $(A, \mathfrak{m})$ is the Hilbert function of $gr_{\mathfrak{m}}(A)$ which is not necessarily level. From now on we say that h is a graded level (resp. Gorenstein) O-sequence if h is the Hilbert function of a level (resp. Gorenstein) graded standard K-algebra. For instance it is well known that the h-vector of a Gorenstein graded K-algebra is symmetric, but this is no longer true for a Gorenstein local ring. Characterizing level O-sequences is a wide open problem in commutative algebra. The problem is difficult and very few results are known even in the graded case as evidenced by [18]. In the following table we give a summary of known results: