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The Galois action and cohomology of a relative homology group of Fermat curves ☆

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For an odd prime p satisfying Vandiver's conjecture, we give explicit formulae for the action of the absolute Galois group $G_{\mathbb{Q}(\zeta_p)}$ on the homology of the degree p Fermat curve, building on work of Anderson. Further, we study the invariants and the first Galois cohomology group which are associated with obstructions to rational points on the Fermat curve.

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1. Introduction

In this paper, we study the action of the absolute Galois group on the homology of the Fermat curve. Let p be an odd prime, let ζ be a chosen primitive pth root of unity, and consider the cyclotomic field $K = \mathbb{Q}(\zeta)$. Let G_K be the absolute Galois group of K. The Fermat curve of exponent p is the smooth projective curve $X \subset \mathbb{P}^2_K$ of genus $g = (p-1)(p-2)/2$ given by the equation

$$x^p + y^p = z^p.$$

Anderson [2] proved several foundational results about the Galois module structure of a certain relative homology group of the Fermat curve. These results are closely related to [13] [7], and were further developed in [1] [3]. Consider the affine open $U \subset X$ given by $z \neq 0$, which has equation $x^p + y^p = 1$. Consider the closed subscheme $Y \subset U$ defined by $xy = 0$, which consists of $2p$ points. Let $H_1(U, Y; \mathbb{Z}/p)$ denote the étale homology group, with $\mathbb{Z}/p$ coefficients, of the pair $(U \otimes \overline{K}, Y \otimes \overline{K})$; it is a continuous module over $G_{\mathbb{Q}}$. There is a $\mu_p \times \mu_p$ action on X given by

$$(\zeta^i, \zeta^j) \cdot [x, y, z] = [\zeta^i x, \zeta^j y, z], \qquad (\zeta^i, \zeta^j) \in \mu_p \times \mu_p,$$

which determines an action on U, preserving Y. By [2, Theorem 6], the group $H_1(U, Y; \mathbb{Z}/p)$ is a free rank one $\mathbb{Z}/p[\mu_p \times \mu_p]$ module, with generator denoted β. The Galois action of $\sigma \in G_K$ is then determined by $\sigma\beta = B_\sigma \beta$, for some unit $B_\sigma \in \mathbb{Z}/p[\mu_p \times \mu_p]$.

Let L be the splitting field of $1 - (1 - x^p)^p$. By [2, Section 10.5], the G_K action on $H_1(U, Y; \mathbb{Z}/p)$ factors through $\mathrm{Gal}(L/K)$. This implies that the full G_K module structure of $H_1(U, Y; \mathbb{Z}/p)$ is determined by the finitely many elements B_q for $q \in \mathrm{Gal}(L/K)$.

From Anderson's work, the description of the elements B_q is theoretically complete in the following sense: Anderson shows that B_q is determined by an analogue of the classical gamma function $\Gamma_q \in \overline{\mathbb{F}}_p[\mu_p] \simeq \overline{\mathbb{F}}_p[\epsilon]/\langle \epsilon^p - 1 \rangle$. By [2, Theorems 7 & 9], there is a formula $B_q = \bar{d}'(\Gamma_q)$ (with $\bar{d}'$ as defined in Section 2.2). The canonical derivation $d : \overline{\mathbb{F}}_p[\mu_p] \to \Omega\overline{\mathbb{F}}_p[\mu_p]$ to the module of Kähler differentials allows one to take the logarithmic derivative $\operatorname{dlog} \Gamma_q$ of Γ_q. Since p is prime, $\operatorname{dlog} \Gamma_q$ determines B_q uniquely [2, 10.5.2, 10.5.3]. The function $q \mapsto \operatorname{dlog} \Gamma_q$ is in turn determined by a relative homology group of the punctured affine line $H_1(\mathbb{A}^1 - V(\sum_{i=0}^{p-1} x^i), \{0, 1\}; \mathbb{Z}/p)$ [2, Theorem 10].

In this paper, for any odd prime p satisfying Vandiver's conjecture, we extend Anderson's work for the Fermat curve of exponent p by finding a closed form formula for B_q