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Stratification of moduli spaces of Lie algebras, similar matrices and bilinear forms ☆

Alice Fialowski [a,b,*], Michael Penkava [c]

[a] University of Pécs, Hungary
[b] Eötvös Loránd University, Hungary
[c] University of Wisconsin–Eau Claire, Eau Claire, WI 54702-4004, United States

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ABSTRACT

In this paper, the authors apply a stratification of moduli spaces of complex Lie algebras to analyzing the moduli spaces of $n \times n$ matrices under scalar similarity and bilinear forms under the cogredient action. For similar matrices, we give a complete description of a stratification of the space by some very simple projective orbifolds of the form $\mathbb{P}^n/G$, where G is a subgroup of the symmetric group Σ_{n+1} acting on $\mathbb{P}^n$ by permuting the projective coordinates. For bilinear forms, we give a similar stratification up to dimension 4.

1. Introduction

The authors have been studying moduli spaces of algebras over the complex numbers for a long time, beginning with the construction of moduli spaces of low dimensional L_∞

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* Corresponding author.
E-mail addresses: fialowsk@ttk.pte.hu, fialowsk@cs.elte.hu (A. Fialowski), penkavmr@uwec.edu (M. Penkava).

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algebras (see, for example [7]). In studying L_∞ and A_∞ algebras, a $\mathbb{Z}_2$-grading plays an important role, but the classical picture of algebras without any $\mathbb{Z}_2$-grading fits into this picture as well.

When analyzing L_∞ algebras on a three dimensional space, we first realized that they are just the ordinary 3-dimensional Lie algebra structures on the space. Because we arrived at our classification of the algebras through an approach that focused on deformations of the algebras, we arrived at a decomposition into strata that had some important differences with the classical decomposition of the space (see [14]). Eventually, we discovered that moduli spaces of low dimensional ordinary and $\mathbb{Z}_2$-graded complex Lie and associative algebras have a decomposition into strata consisting of some very simple types of projective orbifolds, of the form $\mathbb{P}^n/G$, where G is a subgroup of the symmetric group Σ_{n+1}, which acts on $\mathbb{P}^n$ by permuting the projective coordinates.

Based on our construction of many different such moduli spaces, we conjectured that this type of decomposition happens for all such moduli spaces of finite dimensional algebras over $\mathbb{C}$ (see [10,8]). We have verified this conjecture in many cases, but have so far not been able to establish it in general. In this paper, we give an explicit construction of a stratification of part of the moduli space of Lie algebras of a given dimension in exactly this form, which holds in any finite dimensional space.

This part arises when considering Lie algebras which arise as extensions of a 1-dimensional (trivial) Lie algebra by a trivial n-dimensional Lie algebra. These algebras are classified by the action of $\mathbb{C}^* \times \mathbb{GL}(n, \mathbb{C})$ on the space $\mathfrak{gl}(n, \mathbb{C})$ of $n \times n$ matrices by conjugation and multiplication by a scalar, which is sometimes called scalar similarity. We will give a decomposition of the space of equivalence classes of matrices under this action into strata that are parameterized by projective orbifolds. Moreover, the deformations of the elements can be read directly from the forms of the matrices. The classification is related to Jordan decomposition, but is actually coarser, because several classes of Jordan forms combine into one stratum. (In fact, there are a few other differences as well.)

Later, we were asked to compare the algebraic deformation theory with the analytic deformation theory, in particular, to relate the algebraic notion of a miniversal deformation to the analytic one. While these definitions are quite different, the relationship is very close, so we find that our ideas also can be used to stratify moduli spaces arising in the analytic context from the same group action. Arnold in [1] gave a construction of a versal deformation of the moduli space of similar matrices based on their Jordan decomposition. Although the action he considered was similarity, rather than scalar similarity, there is a direct parallel between his analysis and our point of view. Classification of similar matrices was first studied in [1], but has been revisited and improved upon by, for example [13,2,3].

In the construction of moduli spaces, this time of associative algebras, we discovered that a part of the moduli space is described by the cogredient action of $\mathbb{GL}(n, \mathbb{C})$ on $\mathfrak{gl}(n)$, in other words the action given by $G.A = GAG^*$. This moduli space is the space of equivalence classes of bilinear forms on $\mathbb{C}^n$. Bilinear forms over a field of characteristic not