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Direct products of groups and regular orbits

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ABSTRACT

We prove the following result. Let p be a prime, and let G be a finite p-solvable group that is a direct product of two non-cyclic subgroups of coprime order, and let V be some faithful irreducible module for G over some field in characteristic p. Then G has a regular orbit, that is, there exists some $v \in V$ such that $\mathrm{C}_G(v) = 1$.

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1. Introduction

It is well known that the existence of regular orbits has important consequences for the structure of finite groups. The following are examples among many others [11,12,8,14]. Let G be a finite group acting faithfully on a module V. A regular orbit of G on V is any G-orbit in V of size $|G|$. Hence $v \in V$ is in a regular orbit if and only if $\mathrm{C}_G(v) = 1$. The study of conditions on G and V that guarantee the existence of at least one regular orbit has intensively been pursued.

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Important and useful results have been obtained when the module V is taken to be of a particular form. For example, work of Wolf [13], Seress [9] and Dolfi [3] concern modules V that are multiples of a module. Work of Yang [15,16] concerns the case when V is irreducible and quasi-primitive.

For the purpose of this introduction, we focus on the case when G is solvable and V is irreducible. If G is abelian, then V always has regular orbits. When G is nilpotent, this is no longer the case. The issue of determining conditions on G and V that will guarantee the existence of regular orbits was studied by Hargraves [6]. (See also Theorem 3.1 below.) The case when G is supersolvable was studied by Turull [10]. Results on more general kinds of solvable groups have also been obtained, for example [7].

It is a consequence of Hargraves' Theorem 3.1 that if G is nilpotent and two or more Sylow subgroups of G are not cyclic then G has a regular orbit on V. We generalize this result to arbitrary finite p-solvable groups, to say that if G is p-solvable and the direct product of two non-cyclic subgroups of coprime order, then G will have a regular orbit on V.

The following is our main theorem. (See Theorem 2.3 below.)

Theorem. *Suppose that V is a faithful irreducible KG-module where K is a field of characteristic $p \geq 0$, and G is a finite group that is the direct product $G = G_1 \times G_2$ for non-cyclic subgroups G_1 and G_2 of coprime order. If $p > 0$, assume that G is p-solvable. Then G has a regular orbit on V.*

This theorem is the best possible in the following sense. We can not simply replace the condition that G_1 and G_2 be non-cyclic by the condition that G_1 and G_2 be non-trivial. And, we can not remove the condition that G_1 and G_2 be of coprime order. These facts are proved in Section 3 below.

We do not know whether the hypothesis that G be p-solvable when $p > 0$ is needed in our main theorem.

2. The main result

Let G be a finite group acting faithfully on a KG-module V, where K is a field. A *base* for G on V is a subset S of V such that the identity is the only element of G that fixes every element of S, that is $\mathrm{C}_G(S) = 1$. Clearly, any basis of V as a vector space is a base for G. We denote by $b(G)$ the smallest cardinality of a base of G on V. We note that $b(G) \leq \dim_K(V)$, and that, if S is a base for G with cardinality $b(G)$, then S is linearly independent over K. In addition, G has a regular orbit on V if and only if $b(G) \leq 1$.

Recent deep results provide us very strong bounds on the size of bases. They allow us to state the following.

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