Journal of Pure and Applied Algebra 223 (2019) 218–244

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Universal deformation rings and self-injective Nakayama algebras

Frauke M. Bleher [a,*,1], Daniel J. Wackwitz [b]

[a] Department of Mathematics, University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, USA
[b] Department of Mathematics, University of Wisconsin-Platteville, 435 Gardner Hall, 1 University Plaza, Platteville, WI 53818, USA

ARTICLE INFO

Article history:
Received 8 February 2017
Available online 30 March 2018
Communicated by S. Koenig

MSC:
Primary 16G10; secondary 16G20; 20C20

ABSTRACT

Let k be a field and let Λ be an indecomposable finite dimensional k-algebra such that there is a stable equivalence of Morita type between Λ and a self-injective split basic Nakayama algebra over k. We show that every indecomposable finitely generated Λ-module V has a universal deformation ring $R(\Lambda, V)$ and we describe $R(\Lambda, V)$ explicitly as a quotient ring of a power series ring over k in finitely many variables. This result applies in particular to Brauer tree algebras, and hence to p-modular blocks of finite groups with cyclic defect groups.

1. Introduction

Let k be a field of arbitrary characteristic, and let Λ be a finite dimensional algebra over k. Given a finitely generated Λ-module V, it is a natural question to ask over which complete local commutative Noetherian k-algebras R with residue field k the module V can be lifted. Here a lift is a pair (M, ϕ) where M is a free R-module with a Λ-module action and $\phi : k \otimes_R M \to V$ is a Λ-module isomorphism. It was shown in [5, Prop. 2.1] that there exists a particular complete local commutative Noetherian k-algebra $R(\Lambda, V)$ with residue field k and a particular lift (U, ϕ_U) of V over $R(\Lambda, V)$ with the following property: Every lift (M, ϕ) of V over a k-algebra R as above is isomorphic to a specialization of (U, ϕ_U) via a (not necessarily unique) k-algebra homomorphism $R(\Lambda, V) \xrightarrow{\alpha} R$. Moreover, α is unique when $R = k[\epsilon]$ is the ring of dual numbers with $\epsilon^2 = 0$. The ring $R(\Lambda, V)$ is called a versal deformation ring of V and the isomorphism class of the lift (U, ϕ_U) is called a versal deformation of V. One is especially interested in the situation when α is unique for every isomorphism class of lifts of V over *every* k-algebra R as above, and one calls $R(\Lambda, V)$ a universal deformation ring of V in this case. It was shown in [5, Thm. 2.6] that when Λ is self-injective and the stable endomorphism ring of V over Λ is isomorphic to k, then $R(\Lambda, V)$ is universal. The question remains for which

* Corresponding author.
E-mail addresses: frauke-bleher@uiowa.edu (F.M. Bleher), wackwitzd@uwplatt.edu (D.J. Wackwitz).
[1] The first author was supported in part by NSF Grant DMS-1360621.

https://doi.org/10.1016/j.jpaa.2018.03.008

algebras Λ every finitely generated indecomposable non-projective Λ-module has a universal deformation ring.

In this paper, we study the case when Λ is an indecomposable k-algebra that is stably Morita equivalent to a self-injective split basic Nakayama algebra and V is an arbitrary finitely generated indecomposable non-projective Λ-module. Our main goal is to show that no matter how big the k-dimension of the stable endomorphism ring of V is, V always has a universal deformation ring. Moreover, we will give an explicit description of this universal deformation ring for each such V in terms of generators and relations that only depends on the location of $[V]$ in the stable Auslander–Reiten quiver of Λ.

Before stating our results, let us discuss some background on studying lifts and deformation rings of modules.

The problem of lifting modules has a long tradition when Λ is replaced by the group ring kG of a finite (or profinite) group G and k is a perfect field of positive characteristic p. In this case, one not only studies lifts of V to complete local commutative Noetherian k-algebras but to arbitrary complete local commutative Noetherian rings with residue field k. One of the first results in this direction is due to Green who proved in [12] that if k is the residue field of a ring of p-adic integers $\mathcal{O}$ then a finitely generated kG-module V can be lifted to $\mathcal{O}$ if there are no non-trivial 2-extensions of V by itself. Green's work inspired Auslander, Ding and Solberg in [1] to consider more general algebras over Noetherian rings and more general lifting problems. In [19], Rickard generalized Green's result to modules for arbitrary finite rank algebras over complete local commutative Noetherian rings, as a consequence of his study of lifts of tilting complexes. On the other hand, Laudal developed a theory of formal moduli of algebraic structures, and, working over an arbitrary field k, he used Massey products to describe deformations of k-algebras and their modules over complete local commutative Artinian k-algebras with residue field k (see [14] and its references).

Sometimes it may happen that the algebra whose modules and their deformations one would like to study is only known up to a derived or stable equivalence. In [6], the behavior of deformations under such equivalences was studied. In particular, it was shown in [6, Sect. 3.2] that versal deformation rings of modules for self-injective algebras are preserved under stable equivalences of Morita type. Hence these versal deformation rings provide invariants of such equivalences.

In this paper we let k be an arbitrary field, and we concentrate on finite dimensional k-algebras of finite representation type. More specifically, we focus on indecomposable k-algebras Λ for which there exists a stable equivalence of Morita type to a self-injective split basic Nakayama algebra over k.

In [11], Gabriel and Riedtmann showed that Brauer tree algebras are stably equivalent to symmetric split basic Nakayama algebras. Moreover, Rickard proved in [17, Sect. 4] that there is a derived equivalence, and hence by [18, Sect. 5] a stable equivalence of Morita type, between these algebras. Since by [7,10], a p-modular block of a finite group G with cyclic defect groups is a Brauer tree algebra (over a field of characteristic p that is sufficiently large for G), our results apply in particular to this case; see below.

Note that the assumption that Λ is indecomposable is no restriction when one considers deformation rings of finitely generated indecomposable Λ-modules. This follows, since if B is an indecomposable direct factor algebra of Λ and V is a Λ-module that belongs to B then the versal deformation rings of V viewed either as a B-module or as a Λ-module are isomorphic (see Lemma 2.2).

To state our main results, we need the following definition.

Definition 1.1. Let k be a field.

(a) For every positive integer e, let Q_e be the circular quiver with e vertices, labeled $1, \ldots, e$, and e arrows, labeled $\alpha_1, \ldots, \alpha_e$, such that $\alpha_i : i \to i+1$, where the vertex $e+1$ is identified with 1. Let $\mathcal{J}$ be the ideal of the path algebra $k\,Q_e$ generated by all arrows, i.e. by all paths of length 1. For all integers $e \geq 1$ and $\ell \geq 2$, define $\mathcal{N}(e, \ell) = k\,Q_e/\mathcal{J}^\ell$.