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Some algebraic aspects of mesoprimary decomposition

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ABSTRACT

Recent results of Kahle and Miller give a method of constructing primary decompositions of binomial ideals by first constructing "mesoprimary decompositions" determined by their underlying monoid congruences. Monoid congruences (and therefore, binomial ideals) can present many subtle behaviors that must be carefully accounted for in order to produce general results, and this makes the theory complicated. In this paper, we examine their results in the presence of a positive A-grading, where certain pathologies are avoided and the theory becomes more accessible. Our approach is algebraic: while key notions for mesoprimary decomposition are developed first from a combinatorial point of view, here we state definitions and results in algebraic terms, which are moreover significantly simplified due to our (slightly) restricted setting. In the case of toral components (which are well-behaved with respect to the A-grading), we are able to obtain further simplifications under additional assumptions. We also provide counterexamples to two open questions, identifying (i) a binomial ideal whose hull is not binomial, answering a question of Eisenbud and Sturmfels, and (ii) a binomial ideal I for which I_{toral} is not binomial, answering a question of Dickenstein, Miller and the first author.

1. Introduction

A *binomial* is a polynomial with at most two terms; a *binomial ideal* is an ideal generated by binomials. Monomial ideals, well known as objects with rich combinatorial structure, are also binomial. Toric ideals, also of much combinatorial interest, are binomial as well.

Binomial ideals in general are known for being constrained in their geometry and algebra: the irreducible components of a variety defined by binomials (over an algebraically closed field) are toric varieties. More precisely, if the base field is algebraically closed, the associated primes and primary components of binomial

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ideals are binomial (and binomial prime ideals are isomorphic to toric ideals by rescaling the variables). These results form the core of the article [3].

The combinatorial study of binomial primary decomposition was started in [1], but the results in that article require the base field to be algebraically closed of characteristic zero. To completely eliminate assumptions on the base field, a new kind of decomposition, called *mesoprimary decomposition*, from which a primary decomposition can be easily obtained, was introduced in [6]. The main theme in [6] is that the combinatorial structures underlying binomial ideals are *monoid congruences*, that is, equivalence relations on a monoid that are compatible with the additive structure.

The starting point of [6] is that, when performing the primary decomposition of a binomial ideal, the base field plays a role only when one encounters lattice ideals (see Definition 2.3 and Remark 2.9 for details). Indeed, one can decompose a binomial ideal into structurally simpler binomial ideals without regard for the base field. For instance, [3, Theorem 6.2] provides a base field independent decomposition of a binomial ideal into cellular binomial ideals (Definition 2.1) From there, one can proceed, again without field assumptions, to more refined *unmixed decompositions* (see [3, Corollary 8.2] for characteristic zero and [8, Section 4], [4, Theorem 5.1] for results without field assumptions). From unmixed decompositions, one can fairly explicitly obtain primary decompositions. However, unmixed decompositions are not the most refined possible binomial decompositions, nor do they completely reveal the combinatorial structure of the underlying binomial ideal; the mesoprimary decompositions of [6] fulfill those goals.

Monoid congruences (and therefore, binomial ideals) can present many subtle behaviors. To produce general results, these must be carefully accounted for, and this makes the theory complicated. In order to simplify the definitions and results on monoid congruences required to perform mesoprimary decomposition, we restrict our attention in this article to the important class of *positively graded binomial ideals*. Under this assumption, certain pathologies for the corresponding congruences are avoided, and the theory becomes more accessible. Our approach is algebraic: while in [6], key notions are developed first from a combinatorial point of view, here we restate definitions and results in algebraic terms, which are moreover significantly simplified due to our (slightly) restricted setting. These definitions can be found in Section 3, after we review the necessary background on binomial ideals from [3] in Section 2.

The remainder of the paper concerns results and ideas from [1,2] that identify, in the A-homogeneous setting, certain primary components (called *toral* components) that inherit sufficient combinatorial structure from the grading to make them easier to compute. One of the main goals of this project was to obtain analogous methods for computing *toral* mesoprimary components (Definition 4.2). Much to our surprise, the combinatorial methods explored in [1] and [6] appear to be somehow incompatible; each utilizes some underlying combinatorial structure to simplify computation of primary decomposition, but in sufficiently different ways that it is difficult to simultaneously benefit from both outside of highly restricted cases.

Sections 5 and 6 contain some mesoprimary analogs of results from [1] in special cases, as well as examples demonstrating why more general results in this direction are difficult to obtain. We also identify in Example 6.3 a binomial ideal I with the property that the intersection of the toral primary components of I is not a binomial ideal, thus answering a question posed by the authors of [1].

We close this section by addressing a question of Eisenbud and Sturmfels. We recall that the *Hull* of an ideal I, denoted $\mathrm{Hull}(I)$ is the intersection of all the minimal primary components of I. Corollary 6.5 of [3] (see also [4, Theorem 2.10]) states that the Hull of a cellular binomial ideal is binomial.

Problem 6.6 in [3] asks:

Is $\mathrm{Hull}(I)$ binomial for every (not necessarily cellular) binomial ideal I?

We provide a negative answer to this question in the following example.

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