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On the compositum of integral closures of valuation rings ☆

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ABSTRACT

It is well known that if K_1, K_2 are algebraic number fields with coprime discriminants, then the composite ring $A_{K_1}A_{K_2}$ is integrally closed and K_1, K_2 are linearly disjoint over the field of rationals, A_{K_i} being the ring of algebraic integers of K_i. In an attempt to prove the converse of the above result, in this paper we prove that if K_1, K_2 are finite separable extensions of a valued field (K, v) of arbitrary rank which are linearly disjoint over $K = K_1 \cap K_2$ and if the integral closure S_i of the valuation ring R_v of v in K_i is a free R_v-module for $i = 1, 2$ with S_1S_2 integrally closed, then the discriminant of either S_1/R_v or of S_2/R_v is the unit ideal. We quickly deduce from this result that for algebraic number fields K_1, K_2 linearly disjoint over $K = K_1 \cap K_2$ for which $A_{K_1}A_{K_2}$ is integrally closed, the relative discriminants of K_1/K and K_2/K must be coprime.

1. Introduction

For an algebraic number field K, A_K will denote the ring of its algebraic integers. It is well known that if K_1, K_2 are algebraic number fields with coprime discriminants, then the composite ring $A_{K_1}A_{K_2}$ is integrally closed and K_1, K_2 are linearly disjoint over the field $\mathbb{Q}$ of rational numbers (cf. [7, Theorem 4.26], [3, Exercise 4.5.12]). This gives rise to the following natural question:

If K_1, K_2 are algebraic number fields linearly disjoint over $\mathbb{Q}$ for which $A_{K_1}A_{K_2}$ is integrally closed, then is it true that the discriminants of K_1 and K_2 are coprime?

In 2017, we proved that the answer to the above question is in the affirmative when one of K_1 or K_2 is a quadratic field (see [5, Theorem 1.6]). In the present paper we prove that the answer to the above question is always "yes". In this direction, we prove a more general result which will be stated after introducing some notation.

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Notation 1.A. Let R be an integral domain with quotient field K and S be the integral closure of R in a finite separable extension L of K. Assume that S is a free R-module of rank n. As usual the discriminant of S/R to be denoted by $d(S/R)$ is defined to be the ideal in R generated by the determinant of the $n \times n$ matrix $(Tr_{L/K}(\beta_i\beta_j))_{ij}$, where $\{\beta_1, \cdots, \beta_n\}$ is an R-basis of S and Tr is the trace map. As in [7, Proposition 2.9(ii)], it can be easily seen that for any other R-basis $\{\beta'_1, \beta'_2, \cdots \beta'_n\}$ of S the determinants of the matrices $(Tr_{L/K}(\beta_i\beta_j))_{ij}$ and $(Tr_{L/K}(\beta'_i\beta'_j))_{ij}$ differ multiplicatively by a unit. So $d(S/R)$ is well defined.

In this paper, we prove

Theorem 1.1. *Let (K, v) be a valued field of arbitrary rank with perfect residue field and K_1, K_2 be finite separable extensions of K which are linearly disjoint over K. Let S_1, S_2 denote the integral closures of the valuation ring R_v of v in K_1, K_2 respectively. If S_1, S_2 are free R_v-modules and S_1S_2 is integrally closed, then either $d(S_1/R_v)$ or $d(S_2/R_v)$ is the unit ideal.*

The following corollary will be quickly deduced from the above theorem.

Corollary 1.2. *Let K_1, K_2 be algebraic number fields which are linearly disjoint over $K = K_1 \cap K_2$ such that $A_{K_1K_2} = A_{K_1}A_{K_2}$. Then the relative discriminants of the extensions K_1/K and K_2/K are coprime.*

For proving Theorem 1.1, we shall prove the following theorem as a preliminary result. It is of independent interest as well.

Theorem 1.3. *Let $(K, v), K_1, K_2, S_1, S_2$ be as in Theorem 1.1 without the assumption that the residue field of v is perfect. Assume that S_1, S_2 are free R_v-modules and S_1S_2 is integrally closed. If r, s, t denote respectively the number of prolongations of v to K_1, K_2 and K_1K_2, then $t = rs$.*

2. Preliminary results

In what follows for a valuation v of a field K, R_v will denote its valuation ring and M_v the maximal ideal of R_v. (K^h, v^h) will denote the henselization of (K, v) whose valuation ring will be denoted by R^h_v.

The following theorem is already known (see [4, Lemma 2.B, Theorem 2.3]). Its proof is omitted.

Theorem 2.A. *Let (K, v) be a valued field of arbitrary rank with valuation ring R_v and (K^h, v^h) be its henselization having valuation ring R^h_v. Let L be a finite separable extension of K and S be the integral closure of R_v in L. Let $w_1, \cdots, w_t$ be all the prolongations of v to L. Assume that S is a free R_v-module. Then $R^h_{w_i}$ is a free R^h_v-module for $1 \le i \le t$. Moreover one can choose a suitable R^h_v-basis $\mathcal{B}_i \subseteq S$ of $R^h_{w_i}$ such that (i) $\cup_{i=1}^t \mathcal{B}_i$ is an R_v-basis of S; (ii) for every $B_{ij} \in \mathcal{B}_i$ and for each $k \ne i$, $w_k(B_{ij}) \ge v(a) > 0$ for some a in K.*

The proof of the following lemma is contained in the proof of Theorem 1.1 of [4]. For reader's convenience, we prove it here.

Lemma 2.B. *Let $(K, v), R^h_v, L, S, w_1, \cdots, w_t$ and $R^h_{w_i}$ be as in Theorem 2.A. Assume that S is a free R_v-module. Then the R_v-bilinear map from $R^h_v \times S$ into $\prod_{i=1}^t R^h_{w_i}$ mapping (a, α) to $(a\alpha, a\alpha, \cdots, a\alpha)$ for $a \in R^h_v, \alpha \in S$, gives rise to an R^h_v-module isomorphism Λ from $R^h_v \otimes_{R_v} S$ onto $\prod\limits_{i=1}^{t} R^h_{w_i}$.*