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A lower bound for higher topological complexity of real projective space

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ABSTRACT

We obtain an explicit formula for the best lower bound for the higher topological complexity, $\mathrm{TC}_k(RP^n)$, of real projective space implied by mod 2 cohomology.

1. Main theorem

In [2], Farber introduced the notion of topological complexity, $\mathrm{TC}(X)$, of a topological space X. This can be interpreted as one less than the minimal number of rules, called *motion planning rules*, required to tell how to move between any two points of X.[1] This became central in the field of topological robotics when X is the space of configurations of a robot. This was generalized to higher topological complexity, $\mathrm{TC}_k(X)$, of a topological space X by Rudyak in [3]. This can be thought of as one less than the minimal number of rules required to tell how to move consecutively between any k specified points of X ([3, Remark 3.2.7]). In [1], the study of $\mathrm{TC}_k(P^n)$ was initiated, where P^n denotes real projective space. Using $\mathbb{Z}_2$ coefficients for all cohomology groups, define the kth zero-divisor cup-length $\mathrm{zcl}_k(X)$ to be the maximal integer q such that there exist elements $y_1, \ldots, y_q \in \ker(\Delta^* : H^*(X)^{\otimes k} \to H^*(X))$ with nonzero product; i.e., $y_1 \cdots y_q \neq 0$. Here $\Delta : X \to X^k$ is the diagonal map. It is standard ([3, Proposition 3.4] or [1, Proposition 2.2]) that

$$\mathrm{TC}_k(X) \geq \mathrm{zcl}_k(X).$$

In [1, Lemma 5.2], it was shown that

$$\mathrm{zcl}_k(P^n) = \max\{a_1 + \cdots + a_{k-1} : (x_1 + x_k)^{a_1} \cdots (x_{k-1} + x_k)^{a_{k-1}} \neq 0\}$$

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[1] Farber's definition did not include the "one less than" part, but most recent papers have defined it as we are doing here.

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Table 1
Values of $\mathrm{zcl}_k(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\mathrm{zcl}_2(n)$	1	3	3	7	7	7	7	15	15	15	15	15	15	15	15	31	31
$\mathrm{zcl}_3(n)$	2	6	6	12	14	14	14	24	26	30	30	30	30	30	30	48	50
$\mathrm{zcl}_4(n)$	3	8	9	16	19	21	21	32	35	40	41	45	45	45	45	64	67
$\mathrm{zcl}_5(n)$	4	10	12	20	24	28	28	40	44	50	52	60	60	60	60	80	84
$\mathrm{zcl}_6(n)$	5	12	15	24	29	35	35	48	53	60	63	72	75	75	75	96	101
$\mathrm{zcl}_7(n)$	6	14	18	28	34	42	42	56	62	70	74	84	90	90	90	112	118
$\mathrm{zcl}_8(n)$	7	16	21	32	39	48	49	64	71	80	85	96	103	105	105	128	135

in $\mathbb{Z}_2[x_1,\dots,x_k]/(x_1^{n+1},\dots,x_k^{n+1})$. Here $x_i = p_i^*(x)$, where $p_i : (P^n)^k \to P^n$ is projection onto the ith component, and x is the nonzero element of $H^1(P^n)$. Clearly $(x_i + x_k) \in \ker(\Delta^*)$. The goal is to find large nonzero products of powers of these classes.

Our main theorem, Theorem 1.2, gives an explicit formula for $\mathrm{zcl}_k(P^n)$, and hence a lower bound for $\mathrm{TC}_k(P^n)$. It requires the following specialized notation.

Definition 1.1. If $n = \sum \varepsilon_j 2^j$ with $\varepsilon_j \in \{0,1\}$ (so the numbers ε_j form the binary expansion of n), let

$$Z_i(n) = \sum_{j=0}^{i} (1-\varepsilon_j)2^j,$$

and let

$$S(n) = \{i : \varepsilon_i = \varepsilon_{i-1} = 1 \text{ and } \varepsilon_{i+1} = 0\}.$$

Thus $Z_i(n)$ is the sum of the 2-powers $\le 2^i$ which correspond to the 0's in the binary expansion of n. Note that $Z_i(n) = 2^{i+1} - 1 - (n \bmod 2^{i+1})$. The i's in $S(n)$ are those that begin a sequence of two or more consecutive 1's in the binary expansion of n. Also, $\nu(n) = \max\{t : 2^t \text{ divides } n\}$.

Theorem 1.2. *For $n \ge 0$ and $k \ge 3$,*

$$\mathrm{zcl}_k(P^n) = kn - \max\{2^{\nu(n+1)} - 1, 2^{i+1} - 1 - kZ_i(n) : i \in S(n)\}. \tag{1.3}$$

It was shown in [1] that, if $2^e \le n < 2^{e+1}$, then $\mathrm{zcl}_2(P^n) = 2^{e+1} - 1$, which follows immediately from our Theorem 1.6.

In Table 1, we tabulate $\mathrm{zcl}_k(P^n)$ for $1 \le n \le 17$ and $2 \le k \le 8$.

The smallest value of n for which two values of i are significant in (1.3) is $n = 102 = 2^6 + 2^5 + 2^2 + 2^1$. With $i = 2$, we have $7 - k$ in the max, while with $i = 6$, we have $127 - 25k$. Hence

$$\mathrm{zcl}_k(P^{102}) = 102k - \begin{cases} 127 - 25k & 2 \le k \le 5 \\ 7 - k & 5 \le k \le 7 \\ 0 & 7 \le k. \end{cases}$$

For all k and n, $\mathrm{TC}_k(P^n) \le kn$ for dimensional reasons ([1, Prop 2.2]). Thus we obtain a sharp result $\mathrm{TC}_k(P^n) = kn$ whenever $\mathrm{zcl}_k(P^n) = kn$. Corollary 3.4 tells exactly when this is true. Here is a simply-stated partial result.

Proposition 1.4. *If n is even, then $\mathrm{TC}_k(P^n) = kn$ for $k \ge 2^{\ell+1} - 1$, where ℓ is the length of the longest string of consecutive 1's in the binary expansion of n.*

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