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Special unipotent groups are split ☆

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A B S T R A C T

We show that over any field k, a smooth unipotent algebraic k-group is special if and only if it is k-split.

1. Introduction

An *algebraic group* over a field k is a k-group scheme of finite type over k. The smooth affine algebraic k-groups considered here are the same as linear algebraic groups defined over k in the sense of [2]. Recall that an affine algebraic k-group G is called *unipotent* if $G_{\bar{k}}$ (the base change of G to a fixed algebraic closure $\bar{k}$ of k) admits a finite composition series over $\bar{k}$ with each successive quotient isomorphic to a $\bar{k}$-subgroup of the additive group $\mathbb{G}_a$. It is well-known that an affine algebraic k-group G is unipotent if and only if is k-isomorphic to a closed k-subgroup scheme of the group T_n consisting of upper triangular $n \times n$ matrices with all 1 on the main diagonal, for some n.

A smooth unipotent algebraic group G over a field k is called *k-split* if either it is trivial or it admits a composition series by k-subgroups with successive quotients are k-isomorphic to the additive group $\mathbb{G}_a$. We say that G is *k-wound* if every morphism of k-schemes $\mathbb{A}^1_k \to G$ is constant.

Let G be a smooth affine algebraic group over a field k. We say that G is *special*, if for any field extension L/k, every G-torsor over $\operatorname{Spec} L$ is trivial, i.e., if for any field extension L/k, the Galois cohomology set $H^1(L, G)$ is trivial. Special groups have been introduced by Serre in [8]. Over algebraically closed fields, they have been classified by Grothendieck [5].

Suppose that G is a split smooth unipotent group over a field k. By induction on $\dim G$ and the fact that $H^1(L, \mathbb{G}_a) = 0$ for every field extension L over k, we see that G is special. It is also well-known

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that over a perfect field k, every smooth connected unipotent group G is k-split (see e.g. [2, Chapter V, Corollary 15.5 (ii)]). Hence over a perfect field, a smooth unipotent group is special if and only if it is k-split. (Note that a special algebraic group is always connected [8, §4, Theorem 1]. For the reader's convenience, we will provide an alternative proof for this fact when the group is unipotent, see Lemma 2.7.) The following result is the main result of this note.

Theorem 1.1. *Let G be a smooth unipotent algebraic group over a field k. Then G is special if and only if G is k-split.*

Our main result provides an affirmative answer to [9, Question 1.4] on a simple characterization of special unipotent groups over an arbitrary field. M. Huruguen [6] provides a general classification result for special reductive groups over an arbitrary field. He also applies this result to obtain explicit classifications for several classes of special reductive groups such as special semisimple groups, special reductive groups of inner type and special quasisplit reductive groups. R. Achet [1], as a corollary of his main result, proves Theorem 1.1 in the particular case of unipotent groups of dimension 1, by a different method.

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2. Proof of the main result

2.1. Some results of Tits on the structure of unipotent algebraic groups

We first recall some results of Tits concerning the structure of unipotent algebraic groups over an arbitrary (especially imperfect) field of positive characteristic, see [7, Chapter V] and [3, Appendix B].

Let G be a smooth connected unipotent algebraic group over a field k of characteristic $p > 0$. There is a maximal k-split k-subgroup G_s, and it enjoys the following properties: it is normal in G, the quotient G/G_s is k-wound and the formation of G_s commutes with separable (not necessarily algebraic) extensions, see [7, Chapter V, 7] and [3, Theorem B.3.4].

Also there exists a maximal central smooth connected p-torsion k-subgroup of G. This group is called the *cckp-kernel* of G and denoted by $cckp(G)$ or $\kappa(G)$. Here $\dim(\kappa(G)) > 0$ if G is not finite.

The following statements are equivalent:

(1) G is wound over k,
(2) $\kappa(G)$ is wound over k.

If the two equivalent statements are satisfied then $G/\kappa(G)$ is also wound over k ([7, Chapter V, 3.2]; [3, Appendix B, B.3]).

Definition 2.1. Let k be a field of characteristic $p > 0$. A polynomial $P \in k[T_1, \dots, T_r]$ is a *p-polynomial* if every monomial appearing in P has the form $c_{ij}T_i^{p^j}$ for some $c_{ij} \in k$; that is $P = \sum_{i=1}^r P_i(T_i)$ with $P_i(T_i) = \sum_j c_{ij}T_i^{p^j} \in k[T_i]$.

A p-polynomial $P \in k[T_1, \dots, T_r]$ is called *separable* if it contains at least a non-zero monomial of degree 1.

If $P = \sum_{i=1}^r P_i(T_i)$ is a p-polynomial over k in r variables, then the *principal part* of P is the sum of the leading terms of the P_i.