Commuting functions of matrices over topological fields

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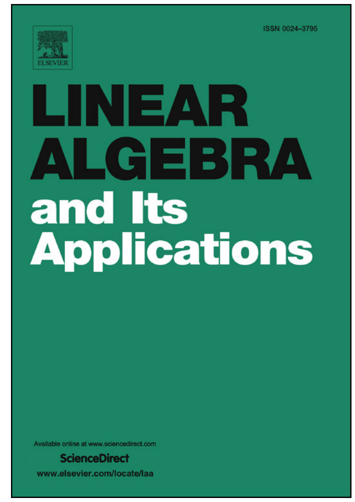

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Abstract

Any continuous function $f : \mathcal{F}^{n\times n} \to \mathcal{F}^{n\times n}$ satisfying $xf(x) \equiv f(x)x$ can be written as $f(x) \equiv \alpha_0(x)x^0 + \ldots + \alpha_{n-1}(x)x^{n-1}$, where $\alpha : \mathcal{F}^{n\times n} \to \mathcal{F}^n$ is a mapping continuous on the set of non-derogatory matrices. Brešar and Šemrl proved this fact for $\mathcal{F} = \mathbb{C}$, and we give a short proof of their result which is moreover valid over any topological field. For $\mathcal{F} \in \{\mathbb{R}, \mathbb{C}\}$, we provide an example of a function f for which any appropriate α is discontinuous at every derogatory matrix.

Keywords: Commuting matrices, double centralizer theorem
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A field $\mathcal{F}$ is called *topological* if it is endowed with a topology $\mathcal{T}$ such that the addition, subtraction, multiplication, and division are continuous. (Here and in what follows, we assume that $\mathcal{F}^n$ holds the natural *product topology*, and we call a mapping $f : X \to Y$ *continuous* if the preimage of any open subset of Y is open in X.) Also, we will assume that any singleton subset of $\mathcal{F}$ is closed and not open; these assumptions are needed to get rid of the cases $\mathcal{T} = 2^{\mathcal{F}}$ and $\mathcal{T} = \{\varnothing, \mathcal{F}\}$, which are trivial to analyze. The obvious fact that every linear mapping on $\mathcal{F}^n$ is continuous implies the following.

Observation 1. *An affine subspace $V \subsetneq \mathcal{F}^n$ is closed and not open.*

1. The result

We recall that a matrix $x \in \mathcal{F}^{n\times n}$ is called non-derogatory if the powers $x^0, \ldots, x^{n-1}$ are linearly independent over $\mathcal{F}$. In other words, there exists a tuple $\tau = (\tau_1, \ldots, \tau_n)$ of indexes in $\{1, \ldots, n\} \times \{1, \ldots, n\}$ such that the

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