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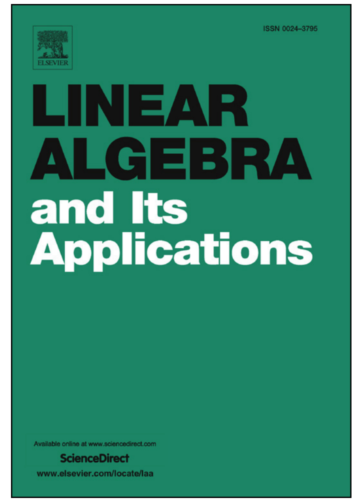

The sum of nonsingular matrices is often nonsingular

József Solymosi

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THE SUM OF NONSINGULAR MATRICES IS OFTEN NONSINGULAR

JÓZSEF SOLYMOSI

ABSTRACT. If $\mathcal{M}$ is a set of nonsingular $k \times k$ matrices then for many pairs of matrices, $A, B \in \mathcal{M}$, the sum is nonsingular, $\det(A+B) \neq 0$. We prove a more general statement on nonsingular sums with a geometric application.

Keywords: sum of nonsingular matrices, polynomial method, geometric removal lemma

1. Introduction

It is a simple fact in linear algebra, that while the product of nonsingular matrices is nonsingular, the similar statement is false for the sum. On the other hand, one can expect some control over such sums since most matrices are nonsingular. (The proper notion of *most matrices* depends on the underlying field. Much more details on this subject can be find in a book of Terry Tao [10].) Finding the inverse or a generalized inverse of the sum of two matrices has important applications in mathematics and in applications. For the review and the history on deriving the inverse of the sum of matrices we refer to [7] and [2]. In this paper we show that—under some mild conditions—if $\mathcal{M}$ is a set of $k \times k$ matrices then for many pairs, $A, B \in \mathcal{M}$, the sum is nonsingular. Our main tool is the so-called "Polynomial Method"[1] which has been used in combinatorics since the 70's and has proven to be very useful in a number of problems. There is a nice lecture book by Larry Guth reviewing old and new applications of the method [6]. There is a striking, very recent application to the cap-set problem by Croot, Lev, and Pach [4] and Ellenberg and Gijswijt [5]. The latter two inspired a large number of interesting results using a counting method similar to what we will apply in this work.

Over finite fields Anderson and Badawi investigated the graph where the vertices are the elements of $SL_n(\mathbb{F}_q)$ and two matrices, $A, B \in SL_n(\mathbb{F}_q)$ form an edge if $\det(A+B) \neq 0$. Akbari, Jamaali and Fakhari [1] proved that the clique number of such graphs is bounded by a universal constant, independent of $\mathbb{F}_q$ for odd q. Their method works over characteristic zero as well. We will refine their result giving almost sharp bound on the clique number. On the other hand, Tomon [12] showed that despite the bounded click number, the chromatic number of this graph increases with q, it is at least $(q/4)^{\lfloor n/2 \rfloor}$.

2. Results

First we state an important case of our main result below. For any set of nonsingular matrices a positive fraction of the pairs add up to a nonsingular matrix.

Theorem 1. *For every $k \in \mathbb{N}$ there is a constant, $c > 0$, depending on k only, such that if $\mathcal{M}$ is a set of nonsingular $k \times k$ matrices over a field K, characteristic $\neq 2$ then the number of pairs, $A, B \in \mathcal{M}$, such that* $\det(A+B) \neq 0$, *is at least* $c|\mathcal{M}|^2$.

We postpone the proof until after our next theorem, where we are going to give a necessary and sufficient condition under which, for a positive fraction of the pairs, $A, B \in \mathcal{M}$, $\det(A+B) \neq 0$. For the exact statement we are going to consider two (not necessary disjoint) sets of $k \times k$ matrices, $\mathcal{M}_1, \mathcal{M}_2$ as the two vertex sets of a bipartite graph, $G(\mathcal{M}_1, \mathcal{M}_2)$, where $A \in \mathcal{M}_1$ and $B \in \mathcal{M}_2$ are connected by an edge iff $\det(A+B) \neq 0$. In what follows we will suppose that $|\mathcal{M}_1| = |\mathcal{M}_2| = n$. A *matching* in a graph is a set of vertex disjoint edges. It follows from elementary graph theory that if the number of

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[1] In many cases the *Linear Algebra Method* or *Rank Method* would be a better description.