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# A note on the Gauss maps of Cayley-free embeddings into spin(7)-manifolds

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ABSTRACT

We show that a closed, orientable 4-manifold $M$ admits a Cayley-free embedding into flat Spin(7)-manifold $\mathbb{R}^8$ if and only if both the Euler characteristic $\chi_M$ and the signature $\tau_M$ of $M$ vanish.



## 1. Introduction

Calibrated geometries, introduced by Harvey and Lawson in [4] are the geometries of minimal submanifolds of a Riemannian manifold $(X, g)$ determined by a special closed differential form $\phi$ on $X$, called a *calibration*. These geometries, especially on spaces with special holonomy, have been the focus of research interests of many geometers and physicists due to their strong relations with gauge theories in higher dimensions [10], mirror symmetry [9] and modern string theory in physics [6]. Hence, understanding their structures plays a key role in these theories.

Harvey and Lawson has recently introduced new tools on calibrated manifolds which can be used to study geometry and analysis. On a calibrated manifold $(X, \phi)$, they define $\phi$-plurisubharmonic functions and $\phi$-convexity which are analogues of classical plurisubharmonic functions and pseudoconvexity in complex

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analysis. They derive many useful results and extend a number of results in complex geometry to calibrated manifolds (cf. [5]).

The analogues in calibrated geometry of totally real submanifolds (ones free of any complex tangent lines) in complex analysis are called $\phi$-free submanifolds and they form a very rich source to construct strictly $\phi$-convex manifolds with every topological type allowed. Similar to totally real submanifolds, there is an integer bound on the possible dimensions of $\phi$-free submanifolds which is called the *free dimension* and is determined by the calibration $\phi$. Furthermore, a strictly $\phi$-convex domain has the homotopy type of a CW complex of dimension at most the free dimension of $\phi$.

In this paper we work on flat Spin(7)-manifold $\mathbb{R}^8$ with Cayley calibration $\Phi$ and investigate the embedding problem of orientable, closed 4-manifolds into $\mathbb{R}^8$ as Cayley-free, which is the only non-trivial case. In [12] we showed that any such manifold with Euler characteristic and signature equal to zero can be embedded into flat $G_2$-manifold $\mathbb{R}^7$ as co-associative-free, hence the inclusion of this embedding into $\mathbb{R}^8$ is automatically Cayley-free. Here we show that the vanishing of these topological invariants is necessary and sufficient for Cayley-free embeddings into $\mathbb{R}^8$ and prove the following theorem.

**Theorem 1.1** *(Main Theorem). A closed, orientable 4-manifold $M^4$ admits a Cayley-free embedding into $\mathbb{R}^8$ if and only if both the Euler characteristic $\chi_M$ and the signature $\tau_M$ of $M$ vanish.*

This theorem actually holds for any flat (or locally flat around a point) Spin(7)-manifold. If $N^8$ is a flat Spin(7)-manifold, then for every point $p \in N$ there is a neighborhood $U_p$ of $p$ which is locally isometric to an open set $V \subseteq \mathbb{R}^8$ and if a closed, orientable 4-manifold $M^4$ admits a Cayley-free embedding into $\mathbb{R}^8$ then it also admits a Cayley-free embedding into $V$ by using a zooming principle similar to coassociative-free embeddings for flat $G_2$-manifolds in [12]. Then, $M^4$ can be embedded into $U_p \subset N^8$ as Cayley-free by using the local isometry. As an example, any closed, orientable 4-manifold $M^4$ with $\chi_M = \tau_M = 0$ can be embedded into flat torus $T^8$ as Cayley-free.

## 2. Background

Let $(X, g)$ be a Riemannian manifold. A closed differential $p$-form $\phi$ with $p \geq 1$ on $X$ is called a *calibration* if

$$\phi|_\xi \leq vol|_\xi$$

for any oriented tangent $p$-plane $\xi$ in $T_xX$ for any $x \in X$. A Riemannian manifold $(X, g)$ with such a differential form $\phi$ is called a *calibrated manifold*. A $p$-plane $\xi$ is called a *$\phi$-plane* or a *calibrated plane* if $\phi|_\xi = vol|_\xi$ and the set of all $\phi$-planes at a point $x$ is called the $\phi$-Grassmannian denoted by $G(\phi_x)$ which is a subset of the oriented Grassmannian $G_p^+(T_xX)$ of oriented $p$-planes in $T_xX$. Furthermore, the set of all $p$-planes on $X$ forms a subbundle of the oriented Grassmannian bundle $\tilde{G}_p(X)$ on $X$.

Each calibration $\phi$ determines a special geometry of submanifolds. An oriented $p$-dimensional submanifold $M^p$ of a calibrated manifold $(X, \phi)$ is called a *calibrated submanifold* or a *$\phi$-submanifold* if each $T_xM \subset T_xX$ lies in $G(\phi_x)$. The fundamental observation about these submanifolds, which is also called *the fundamental theorem of calibrated geometry*, is that all $\phi$-submanifolds are absolutely volume-minimizing in their homology class.

There are lots of interesting examples of calibrations with rich geometries, especially on manifolds with special holonomy. Kähler, Calabi–Yau, hyper-Kähler, $G_2$ or Spin(7) manifolds come up with one or more canonical calibrations. In this paper we are just interested in flat Spin(7)-manifold $\mathbb{R}^8$ with Cayley calibration, for other examples of calibrations one may look at the foundational paper [4] by Harvey and Lawson or the recent monograph [6] by Joyce, which is mainly focusing on the manifolds with special holonomy.