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Comparing curves in homogeneous spaces

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ABSTRACT

Of concern is the study of the space of curves in homogeneous spaces. Motivated by applications in shape analysis we identify two curves if they only differ by their parametrization and/or a rigid motion. For curves in Euclidean space the Square-Root-Velocity-Function (SRVF) allows to define and efficiently compute a distance on this infinite dimensional quotient space. In this article we present a generalization of the SRVF to curves in homogeneous spaces. We prove that, under mild conditions on the curves, there always exist optimal reparametrizations realizing the quotient distance and demonstrate the efficiency of our framework in selected numerical examples.

1. Introduction

Comparing shapes of curves is a topic of intrinsic interest and, in addition, it is of relevance in many applications in the broad area of shape analysis [39,33,5]. Usually the notion of "shape" means comparing curves without regard to rigid motions or reparametrizations. Thus, it implies modding out the space of parametrized curves by the group of rigid motions, and/or the group of reparametrizations. We might be interested in curves in a flat Euclidean space (for example, the outline of an image in a photograph), or we might be interested in curves that lie on a space that is itself curved (for example, hurricane tracks on the surface of the earth or paths of positive definite symmetric matrices in brain connectivity analysis). This paper is primarily concerned with the second of these two cases.

To outline our approach to this problem, let $\mathcal{P}([0,1],M)$ denote the set of parametrized curves in a Riemannian manifold M. Thinking of $\mathcal{P}([0,1],M)$ as an infinite dimensional manifold, we wish to equip

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it with a Riemannian metric that is invariant under the group of isometries of M and under the group of reparametrizations. In this way, we can induce a metric on the quotient of $\mathcal{P}([0,1], M)$ by either, or both, of these groups. This will allow us to quantify the difference between shapes of curves by calculating the length of the shortest geodesic joining them in the quotient space. We can also perform statistical analyses on sets of curves by using techniques of non-linear statistics on this quotient manifold.

For the case $M = \mathbb{R}^n$ several metrics have been defined satisfying the required invariances, see, e.g., [5,6,25,34,32,37,21] and the references therein. The main goal of this paper is to take a particularly useful one of these metrics, the elastic metric associated with the "square root velocity function" (SRVF), and generalize it to curves in a homogeneous manifold M. (A homogeneous manifold is a quotient of a Lie group by a closed subgroup. For the purpose of this paper we will only consider the case where the subgroup is compact.)

Previous work on curves in $\mathbb{R}^n$: In [26,24,2], Michor and Mumford showed that the simplest reparametrization invariant L^2-metric on $\mathcal{P}([0,1], \mathbb{R}^n)$ is an inadequate choice for shape analysis as it results in vanishing geodesic distance, i.e., for any two curves $c_1, c_2 \in \mathcal{P}([0,1], \mathbb{R}^n)$ there exist paths of arbitrarily short length connecting them. Subsequently it has been shown in [25] that this degeneracy can be overcome by adding higher order derivatives in the definition of the metric, yielding to the class of reparametrization invariant Sobolev metrics. While this class of metrics allows one to prove strong theoretical results [10], it can be difficult to calculate the corresponding minimizing geodesics and thus obtain the distance function on the shape space of curves. (See also the recent article on a numerical framework for general second order Sobolev metrics [3].)

For planar curves (i.e., $M = \mathbb{R}^2$), Younes et al. [40,38] consider a specific first order Sobolev metric, that gives rise to an efficient method for calculating geodesics in the space of parametrized curves. Their methods are, however, very specific to $\mathbb{R}^2$.

In [27], Mio et al. considered a family of "elastic metrics" on the space of planar curves. Intuitively, this family allows one to attach different weights to perturbations in the tangent direction ("stretching") and in the normal direction ("bending"). A precise formula for this metric is given by

$$G_c(v_1, v_2) = \int_0^1 a^2 \langle D_s v_1^\perp, D_s v_2^\perp \rangle + b^2 \langle D_s v_1^\top, D_s v_2^\top \rangle ds, \tag{1}$$

where $c : [0,1] \to \mathbb{R}^2$ is a parametrized curve, v_1 and v_2 are vector fields along this curve, D_s and ds denote differentiation and integration with respect to arc-length, and $D_s v_1^\perp$ (resp. $D_s v_1^\top$) denotes the component of $D_s v_1$ that is normal (resp. tangent) to the tangent vector c' of the curve. For the case $a = b$, this metric is precisely the one studied by Younes et al.

In [34], Srivastava et al. found, analogous to the transformation of [40], an efficient representation of the elastic metric with parameter values $a = 1$ and $b = \frac{1}{2}$. In contrast to the work [40] their framework is valid for curves with values in arbitrary $\mathbb{R}^n$. This method, known as the square root velocity function, has proved extremely successful for computations and has been used in numerous applications in shape analysis, see [33] and the references therein. The SRVF method has several important properties:

1. The metric is extended to the space of all absolutely continuous curves, a much larger space of curves than smooth immersions.
2. The space of open parametrized curves is metrically and geodesically complete, and there are explicit formulas to compute geodesics.
3. As a consequence of 2, modding out by the reparametrization group can be implemented efficiently using, e.g., a dynamic programming algorithm.