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On a class of Einstein-reversible Finsler metrics

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ABSTRACT

In this paper, we introduce the notion of Einstein-reversibility for Finsler metrics. We study a class of p-power Finsler metrics defined by a Riemann metric and 1-form which are Einstein-reversible. It shows that such a class of Einstein-reversible Finsler metrics are always Einstein metrics. In particular, it indicates that all p-power metrics but Randers metrics, square metrics and 2-dimensional square-root metrics, are always Ricci-flat-parallel. Further, the local structure is determined for 2-dimensional square-root metrics which are Einsteinian, and such metrics are not necessarily Ricci-flat.

1. Introduction

In Finsler geometry, the Finsler metrics under consideration might not be reversible. This leads to the non-reversibility of geodesics and curvatures. However, for certain non-reversible Finsler metrics, the geodesics are possibly reversible; if the geodesics are non-reversible, the curvatures might be reversible. M. Crampin has shown that a Randers metric $F = \alpha + \beta$ has strictly reversible geodesics if and only if β is parallel ([6]). Later, Masca–Sabau–Shimada studied (α, β)-metrics with reversible geodesics ([7] [8]). In [12], the present author and Z. Shen introduce a weaker reversibility than geodesic-reversibility and they study Randers metrics with reversible Riemann curvature and Ricci curvature.

In this paper, we study the reversibility of Einstein scalar in Finsler geometry. For a Finsler metric $F = F(x, y)$ on a manifold M, the Riemann curvature $R_y : T_xM \to T_xM$ is a family of linear transformations and the Ricci curvature $Ric(x, y) = \text{trace}(R_y)$, $\forall y \in T_xM$. We can always express the Ricci curvature $Ric(x, y)$ as $Ric(x, y) = (n - 1)\lambda(x, y)F^2$ for some scalar $\lambda(x, y)$ on TM, where $\lambda(x, y)$ is called the *Einstein scalar*. A Finsler metric F is said to be *Einstein-reversible* if the Einstein scalar is reversible, namely, $\lambda(x, y) = \lambda(x, -y)$. Note that Ricci-reversibility does not imply Einstein-reversibility, and Einstein-reversibility does not imply Ricci-reversibility. Clearly, F is Einstein-reversible if F is reversible.

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If the Einstein scalar $\lambda(x,y)$ is a scalar on M, namely, $\lambda(x,y) = \lambda(x)$, then F is called an *Einstein metric*. Einstein metrics are a natural extension of those in Riemann geometry and they have been shown to have similar good properties as in Riemann geometry for some special Finsler metrics ([1] [3] [4] [11] [14]). It is clear that an Einstein metric is Einstein-reversible. If a Finsler metric F is of scalar flag curvature, then F is Einstein-reversible iff. F has reversible flag curvature. In particular, in two-dimensional case, Einstein-reversibility is equivalent to the reversibility of the flag curvature. It should be an interesting point to study Finsler metrics which are Einstein-reversible. This is a difficult problem for general Finsler metrics. We shall restrict our attention to a class of p-power (α,β) Finsler metrics in the following form

$$F = \alpha(1 + \frac{\beta}{\alpha})^p, \tag{1}$$

where $p \neq 0$ is a real constant, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is Riemannian and $\beta = b_i(x)y^i$ is a 1-form.

In (1), if $p = 1$, then $F = \alpha+\beta$ satisfying $b := \|\beta\|_\alpha < 1$ is called a *Randers metric*. Randers metrics with special curvature properties have been studied by many people in recent years. If $p = 2$, then $F = (\alpha+\beta)^2/\alpha$ satisfying $b := \|\beta\|_\alpha < 1$ is called a *square metric*. Square metrics have also been shown to have some special geometric properties ([3] [13] [15] [19]). If $p = -1$, then $F = \alpha^2/(\alpha+\beta)$ satisfying $b := \|\beta\|_\alpha < 1/2$ is called a *Matsumoto metric* introduced by M. Matsumoto in [9]. If $p = 1/2$, then $F = \sqrt{\alpha(\alpha+\beta)}$ satisfying $b := \|\beta\|_\alpha < 1$ is called a *square-root metric*, and a two-dimensional square-root metric has some special properties (see Theorem 1.2 below). The four cases are special and will be singled out in the proof of Theorem 1.1 below.

Theorem 1.1. *Let F be a p-power (α,β)-metric defined by (1) on an n-dimensional manifold. Then F is Einstein-reversible if and only if F is an Einstein metric. Further, except for $p = 1, 2$, and $p = 1/2$ but $n = 2$, F is Ricci-flat-parallel (α is Ricci-flat and β is parallel with respect to α).*

It is shown in [12] that the Ricci curvature of a Randers metric is reversible if and only if the Ricci curvature is quadratic. Theorem 1.1 shows a similar property for p-power metrics of Einstein-reversibility. We have not found a non-trivial Einstein-reversible metric.

For the further geometric structure, it proves that any $n(\geq 3)$-dimensional Einstein Randers metric is of Ricci constant and in particular is of constant flag curvature when $n = 3$ ([1]); the local structure of an Einstein square metric can be determined up to the local structure of an Einstein Riemann metric (see [3]). For a two-dimensional square-root metric of Einstein-reversibility, the local structure can be determined by the following theorem.

Theorem 1.2. *Let $F = \sqrt{\alpha(\alpha+\beta)}$ be a two-dimensional square-root metric which is Einsteinian (equivalently, of isotropic flag curvature). Then α and β can be locally written as*

$$\alpha = \frac{\sqrt{B}}{(1-B)^{\frac{3}{4}}}\sqrt{\frac{(y^1)^2+(y^2)^2}{u^2+v^2}}, \qquad \beta = \frac{B}{(1-B)^{\frac{3}{4}}}\frac{uy^1+vy^2}{u^2+v^2}, \tag{2}$$

where $0 < B = B(x) < 1, u = u(x), v = v(x)$ are some scalar functions which satisfy the following PDEs:

$$u_1 = v_2, \quad u_2 = -v_1, \quad uB_1 + vB_2 = 0, \tag{3}$$

where $u_i := u_{x^i}, v_i := v_{x^i}$ and $B_i := B_{x^i}$. Further, the isotropic flag curvature K is given by

$$K = -\frac{(u^2+v^2)\sqrt{1-B}}{2B^2}(B_{11}+B_{22}) - \frac{(u^2+v^2)^2(3B-2)}{4B^3\sqrt{1-B}}\left(\frac{B_1}{v}\right)^2, \tag{4}$$

where $B_{ij} := B_{x^ix^j}$.