Differential Geometry and its Applications 59 (2018) 204–226

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Deformations of Courant algebroids and Dirac structures via blended structures

Xiang Ji

The Pennsylvania State University, New Kensington 3550 Seventh Street Road, New Kensington, PA 15068, United States

ARTICLE INFO

Article history:
Received 6 January 2018
Received in revised form 12 May 2018
Available online 16 May 2018
Communicated by J. Stasheff

ABSTRACT

Deformations of a Courant algebroid $(E, \langle\cdot,\cdot\rangle, \circ, \rho)$ and its Dirac subbundle A have been widely considered under the assumption that the pseudo-Euclidean metric $\langle\cdot,\cdot\rangle$ is fixed. In this paper, we attack the same problem in a setting that allows $\langle\cdot,\cdot\rangle$ to deform. Thanks to Roytenberg, a Courant algebroid is equivalent to a symplectic graded Q-manifold of degree 2. From this viewpoint, we extend the notions of graded Q-manifold, DGLA and L_∞-algebra all to "blended" versions to combine two differentials of degree ± 1 together, so that Poisson manifolds, Lie algebroids and Courant algebroids are unified as blended Q-manifolds; and define a submanifold $\mathcal{A}$ of "coisotropic type" which naturally generalizes the concepts of coisotropic submanifolds, Lie subalgebroids and Dirac subbundles. It turns out the deformations of a blended homological vector field Q are controlled by a blended DGLA, and the deformations of $\mathcal{A}$ are controlled by a blended L_∞-algebra. The results apply to the deformations of a Courant algebroid and its Dirac structures, the deformations of a Poisson manifold and its coisotropic submanifold, and the deformations of a Lie algebroid and its Lie subalgebroid.

1. Introduction

Deformation theories are traditionally described using differential graded Lie algebras (DGLA, for short), with the goal of establishing the 1–1 correspondence between the deformations of a given object and the Maurer–Cartan elements of an associated DGLA. However, some deformation problems require an extension of the concept so that certain higher brackets are involved in the Maurer–Cartan equation. Such a notion is called an L_∞-algebra, and nowadays it is widely adopted for deformations. An L_∞-algebra L is said to control the deformations of an object if the deformations are exactly those solutions of the Maurer–Cartan equation of L. Therefore, given a deformation problem in consideration, a routine task is to construct an L_∞-algebra and show it controls the deformations.

E-mail address: xxj104@psu.edu.

https://doi.org/10.1016/j.difgeo.2018.05.002
0926-2245/

The motivating problem of this paper is the deformation problems of a Courant algebroid and its Dirac structures. The Courant bracket was first studied by T. Courant ([2]), and was later formalized by Z. Liu, A. Weinstein and P. Xu to study the structure of Lie bialgebroids. Roughly speaking, a Courant algebroid is a vector bundle E equipped with a pseudo-inner product $\langle\cdot,\cdot\rangle$, an anchor ρ from E to the tangent bundle TM of the base manifold M, and a bracket $\circ$ on the space $\Gamma(E)$ that satisfies the Jacobi identity but fails to be antisymmetric. A Dirac structure is an involutive Lagrangian subbundle A of E. These concepts lend great power in unifying geometric structures such as symplectic, Poisson, complex and generalized complex structures.

Due to the wide applications of Courant algebroid and Dirac structure, their deformation problems have been broadly considered and keep attracting interest. In [11], Z. Liu, A. Weinstein and P. Xu first construct a DGLA $(\Omega_A, d_A, [\cdot,\cdot]_L)$ associated to the Dirac structure A assuming the existence of a transversal Dirac structure L, and use it to control the small deformations of A. Later in [8], formal deformations of a Dirac structure are considered. Concerning Courant algebroid E, from a purely algebraic viewpoint, the deformations are described via Poisson algebra in [9] under the assumption that the pseudo-inner product is fixed. With the same assumption, in [5], via D. Roytenberg's derived brackets viewpoint, a DGLA is constructed to control the deformations of the bracket $\circ$ and the anchor ρ; upon the selection of a complement B, an L_∞-algebra $V_{A,B}$ is constructed to control the deformations of A; the two are combined to control their simultaneous deformations. Very recently, the uniqueness of the DGLA $(\Omega_A, d_A, [\cdot,\cdot]_L)$ in [11] and the L_∞-algebra $V_{A,B}$ in [5] is proved in [6]. To be specific, a different selection of the complement results in an L_∞-isomorphic DGLA or L_∞-algebra.

In this paper, we attempt to attack the deformation problem of Courant algebroid and Dirac structure in the most general setup:

1) the pseudo-inner product $\langle\cdot,\cdot\rangle$ is deformed with the bracket $\circ$ and the anchor ρ;

2) a deformed subbundle of A is not assumed to be Lagrangian in advance.

Thanks to D. Roytenberg ([15]), Courant algebroids are in 1–1 correspondence with symplectic graded Q-manifolds of degree 2, i.e. a graded vector bundle $\mathcal{E}$ equipped with a symplectic structure Ω of degree 2 and an involutive vector field X_Q of degree 1 that preserves Ω. Considering the Poisson structure π inverse to Ω, by carefully selecting the degree on multi-vector fields, π is of degree -1. The pair (π, X_Q) is equivalent to the Courant algebroid structure $(\langle\cdot,\cdot\rangle, \circ, \rho)$ on E and satisfies

$$\begin{cases} [\pi,\pi]_{\mathrm{SN}} = 0, \\ [\pi, X_Q]_{\mathrm{SN}} = 0, \\ [X_Q, X_Q]_{\mathrm{SN}} = 0. \end{cases}$$

Here $[\cdot,\cdot]_{\mathrm{SN}}$ is the Schouten–Nijenhuis bracket of multi-vector fields on graded manifolds. Let $\mathfrak{X}(\mathcal{E})$ be the collection of multi-vector fields on $\mathcal{E}$. It is clear that the operator $[X_Q,\cdot]$ is a differential of the DGLA $(\mathfrak{X}(\mathcal{E}), [\cdot,\cdot]_{\mathrm{SN}})$. However, although the composition of the operator $[\pi,\cdot]$ with itself is zero, it is a differential of degree -1 instead of $+1$. In order to combine X_Q and π together, we are motivated to define the notion of a 'blended' differential, which is a sum of two compatible differentials of degree ± 1. Following this idea, the notion of DGLA is extended to a blended version, i.e. a graded Lie algebra with a blended differential, which is then used to control the deformations of a Courant algebroid.

On the other hand, the vector field $\pi + X_Q$ satisfies $[\pi + X_Q, \pi + X_Q]_{\mathrm{SN}} = 0$, which is analogous to a homological vector field. With this idea in mind, we immediately notice a list of geometric structures that fall into the same scheme.