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# Closed geodesics with local homology in maximal degree on non-compact manifolds

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ABSTRACT

We show that, on a complete and possibly non-compact Riemannian manifold of dimension at least 2 without close conjugate points at infinity, the existence of a closed geodesic with local homology in maximal degree (i.e. in degree index plus nullity) and maximal index growth under iteration forces the existence of infinitely many closed geodesics. For closed manifolds, this was a theorem due to Hingston.


## 1. Introduction

Since the end of the 19th century, with the work of Hadamard [18] and Poincaré [32,31], the study of closed geodesics in Riemannian or Finsler manifolds has occupied a central role in geometry and dynamics, and has motivated the developments of sophisticated tools in non-linear analysis (Morse theory, to mention a particularly successful one) and symplectic geometry. The intrigue in the theory comes from the fact that closed geodesics, although are often abundant in closed manifolds, are nevertheless hard to find. While the existence of one closed geodesic [6,25] is a consequence of a simple (for nowadays standards) minimax trick, the existence of a second closed geodesic on a simply connected closed Riemannian manifold is a difficult problem that has been settled only in certain cases, see [8,24] and references therein. The celebrated closed geodesics conjecture claims that every closed Riemannian manifold of dimension at least 2 possesses infinitely many closed geodesics. Currently, the conjecture is known for large classes of closed manifolds [16,5,3,11],

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but is still widely open for instance for higher dimensional spheres, complex projective spaces, and general compact rank-one symmetric spaces. Evidence supporting the closed geodesics conjecture is given by a theorem due to Rademacher [33,34], which confirms the conjecture for a $C^2$-generic Riemannian metric on any given simply connected closed Riemannian manifold of dimension at least 2.

On general complete, but non-compact, Riemannian manifolds there may be only finitely many closed geodesics (e.g. in cylinders with constant negative curvature), and sometimes even no closed geodesics at all (e.g. in flat Euclidean spaces). From the variational perspective, the ends of the manifold cause a lack of compactness for the sublevel sets of the energy function, and therefore the standard techniques from non-linear analysis fail to provide critical points. In 1992, Benci and Giannoni [4] introduced a penalization method and employed it to show the existence of a closed geodesic on every complete Riemannian manifold with a certain minimum of loop space homology, and under a suitable condition on the geometry at infinity: the Riemannian metric must have non-positive curvature at infinity. In the recent paper [1], the authors improved Benci and Giannoni's result as follows: every $d$-dimensional complete Riemannian manifold *without close conjugate points at infinity* has at least a closed geodesic provided its free loop space homology is nontrivial in some degree larger than $d$, and infinitely many closed geodesics provided the set of Betti numbers of its free loop space homology in degrees larger than $d$ is unbounded. The condition of not having close conjugate points at infinity means that, for every length $\ell > 0$, there exists a compact subset of the manifold outside which no pairs of points are conjugate along geodesics of length less than or equal to $\ell$.

On closed Riemannian manifolds of dimension at least 2, Hingston [20] showed that the presence of a particular kind of closed geodesic – one with local homology in maximal degree and maximal index growth under iteration – forces the existence of infinitely many more closed geodesics. This remarkable and hard theorem is one of the crucial ingredients for proving that the growth rate of closed geodesics on any Riemannian 2-sphere is at least the one of prime numbers. Our main result extends the validity of Hingston's theorem to non-compact complete Riemannian manifolds without close conjugate points at infinity.

**Theorem 1.1.** *Let $(M, g)$ be a complete Riemannian manifold of dimension at least 2 without close conjugate points at infinity and possessing a non-iterated closed geodesic $\gamma$ such that:*

(i) *the local homology $\mathrm{C}_d(E, \gamma)$ is non-trivial in degree $d = \overline{\mathrm{ind}}(\gamma) + \dim(M) - 1$;*
(ii) $\mathrm{ind}(\gamma^m) + \mathrm{nul}(\gamma^m) = m\,\overline{\mathrm{ind}}(\gamma) + \dim(M) - 1$ *for all $m \in \mathbb{P} \cup \{1\}$, where $\mathbb{P} \subset \mathbb{N}$ is some infinite subset of prime numbers.*

*Then $(M, g)$ contains infinitely many closed geodesics.*

We refer the reader to Sections 2.2 and 3 for the background on the Morse index and the local homology of closed geodesics. Hingston showed that the above conditions (i–ii) imply that each iterate $\gamma^m$, for $m \in \mathbb{P} \cup \{1\}$, has non-trivial local homology in maximal degree $\mathrm{ind}(\gamma^m) + \mathrm{nul}(\gamma^m)$, and trivial local homology in all other degrees. This will be explained with full details also in Section 5.1. One can also show that the closed geodesic $\gamma$ must be degenerate, and indeed must have only 1 as Floquet multiplier.

An earlier result of Bangert and Klingenberg [7, Theorem 3], along the line of Hingston's one, stated that a closed Riemannian manifold $(M, g)$ has infinitely many closed geodesics every time it has one, say $\gamma$, with average index $\overline{\mathrm{ind}}(\gamma) = 0$, non-trivial local homology, and that is not a global minimizer of the energy function in its connected component. This theorem has an intersection with Hingston's one: the two results cover the case in which the special closed geodesic $\gamma$ has average index $\overline{\mathrm{ind}}(\gamma) = 0$ and non-trivial local homology $\mathrm{C}_d(E, \gamma)$ in degree $d = \dim(M) - 1 \geq 1$. Remarkably, our argument for proving Theorem 1.1 does not allow to extend the remaining cases in Bangert and Klingenberg's full theorem (that is, when $d \in \{0, ..., \dim(M) - 2\}$) to general non-compact manifolds without close conjugate points at infinity.