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The Morse index of a triply periodic minimal surface

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ABSTRACT

In the previous work, the first author established an algorithm to compute the Morse index and the nullity of an n-periodic minimal surface in $\mathbb{R}^n$. In fact, the Morse index can be translated into the number of negative eigenvalues of a real symmetric matrix and the nullity can be translated into the number of zero-eigenvalue of a Hermitian matrix. The two key matrices consist of periods of the abelian differentials of the second kind on a minimal surface, and the signature of the Hermitian matrix gives a new invariant of a minimal surface. On the other hand, H family, rPD family, tP family, tD family, and tCLP family of triply periodic minimal surfaces in $\mathbb{R}^3$ have been studied in physics, chemistry, and crystallography. In this paper, we first determine the two key matrices for the five families explicitly. As its applications, by numerical arguments, we compute the Morse indices, nullities, and signatures for the five families.

1. Introduction

The Morse index (resp. the nullity) of a compact oriented minimal submanifold in an oriented Riemannian manifold is defined as the sum of the dimensions of the eigenspaces corresponding to negative eigenvalues (resp. zero-eigenspace) of the Jacobi operator of the area. The purpose of this work is to compute the Morse index and the nullity of a compact oriented minimal surface in a flat three-torus. Combining our results and the scheme given in [9] yields the existence of triply periodic constant mean curvature surfaces with small constant mean curvature. Moreover, we consider a signature of a minimal surface defined in [3], and compute the signature of a compact oriented minimal surface in a flat three-torus. Now we refer to backgrounds.

In 1968, Simons [20] gave the second variational formula of the area and compute the Morse index and the nullity of a totally geodesic subsphere in the sphere. By his technique, we can see that the Morse index

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(resp. the nullity) of a totally geodesic subtorus in a flat three-torus is zero (resp. one). Next impressive developments were obtained by Montiel–Ros [15] and Ross [17]. Montiel–Ros considered the Dirichlet eigenvalue problem and the Neumann eigenvalue problem of the Laplacian to compute the Morse index and the nullity of a minimal surface. Their study includes the result that tCLP family consists of minimal surfaces with Morse index three and nullity three. Ross proved that Schwarz P surface, D surface, and Schoen's Gyroid are volume preserving stable, respectively. Applying his arguments, we find that each of the three minimal surfaces has Morse index one and nullity three. But the Morse index and the nullity have not been computed for other examples in the past two decades.

Recently, the first author [2,3] established a Moduli theory of compact oriented minimal surfaces in flat tori via the Morse index and the nullity. Also, he gave a procedure to compute the Morse index of a minimal surface with only trivial Jacobi fields. Recall that a normal vector field vanishing the Jacobi operator of the area is called a *Jacobi field* and the dimension of the space of Jacobi fields is equal to the nullity of a minimal surface. It is well-known that normal components of the Killing vector fields generated by translation on the torus give rise to Jacobi fields. So if we consider a non-totally geodesic compact oriented minimal surface in a flat n-torus $\mathbb{R}^n/\Lambda$, then it has nullity at least $n(=\dim \mathbb{R}^n/\Lambda)$. The Killing vector fields generated by translation on the torus is called *trivial Jacobi fields*, and a minimal surface has only trivial Jacobi fields if and only if its nullity is equal to n. The procedure is to reduce computing the Morse index and the nullity of such minimal surfaces to fundamental arguments of eigenvalues in linear algebra. More precisely, the Morse index can be translated into the number of negative eigenvalues of a real symmetric matrix (see $W_2 - W_1$ in Section Two) and the nullity can be translated into the number of zero-eigenvalue of a Hermitian matrix (see W in Section Two). The two key matrices are derived from the period matrix given by the abelian differentials of the second kind (meromorphic differentials with zero residues) on a minimal surface (see Theorems 3.2, 3.4, 3.6, 3.8), and the signature of W gives a new invariant of a minimal surface. The present work suggests that it might be easier to compute the signature than the Morse index. The main difficulty is to determine a canonical homology basis and the period matrix for each minimal surface explicitly (see Theorems 3.1, 3.3, 3.5, 3.7). Our technique developed in this paper is to overcome this by using the method which is faithful to the basics of compact Riemann surfaces, and shows that the procedure turns out to be practical.

On the other hand, triply periodic minimal surfaces in $\mathbb{R}^3$ have been studied in physics, chemistry, and crystallography. Schröder-Turk, Fogden, and Hyde [21] studied one-parameter families of triply periodic minimal surfaces in $\mathbb{R}^3$. These one-parameter families which are called H family, rPD family, tP family, tD family, and tCLP family, contain many classical examples (Schwarz P surface, D surface, Schwarz H surface, and Schwarz CLP surface). Note that a triply periodic minimal surface properly immersed in $\mathbb{R}^3$ corresponds to a minimal immersion of a compact oriented surface into a flat three-torus. Hence the above one-parameter families are related to our works. Also, the above five families consist of minimal surfaces of genus three, and the following Ros' result clarifies an importance of such families. In fact, in this case, the minimum value of Morse index must be one (see Remark 4.1) and Ros [16] proved that genus of a compact oriented minimal surface in a flat three-torus with Morse index one must be three. In the present paper, we compute the Morse index, the nullity, and the signature of each of the above one-parameter families as numerical applications of our theorems.

To state them, we review some fundamental arguments in the theory of minimal surfaces. Let $f : M \to \mathbb{R}^n/\Lambda$ be a minimal immersion of a compact oriented surface M into a flat n-torus $\mathbb{R}^n/\Lambda$. With the induced conformal structure, M is a compact Riemann surface and f is called a *conformal minimal immersion*. Our object is then to study conformal minimal immersions of compact Riemann surfaces in flat n-tori. For a conformal minimal immersion, the following theorem is a basic tool.

Theorem 1.1 *(Weierstrass representation formula). Let $f : M \to \mathbb{R}^n/\Lambda$ be a conformal minimal immersion. Then, up to translations, f can be represented by the following path-integrals*: