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Various expansive measures for flows ☆

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Abstract

We discuss a characterization of countably expansive flows in measure-theoretical terms as in the discrete case [2]. More precisely, we define the *countably expansive flows* and prove that a homeomorphism of a compact metric space is countable expansive just when its suspension flow is. Moreover, we exhibit a measure-expansive flow (in the sense of [4]) which is not countably expansive. Next we define the weak expansive measures for flows and prove that a flow of a compact metric space is countable expansive if and only if it is *weak measure-expansive* (i.e. every orbit-vanishing measure is weak expansive). Furthermore, unlike the measure-expansive ones, the weak measure-expansive flows may exist on closed surfaces. Finally, it is shown that the integrated flow of a C^1 vector field on a compact smooth manifold is C^1 stably expansive if and only if it is C^1 stably weak measure-expansive.

MSC: primary 54H20; secondary 37B05

Keywords: Expansive homeomorphism; Expansive measure; Countably expansive

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1. Introduction

The expansive homeomorphisms were conceived by Utz in the middle of the twenty century [10]. Precisely, a homeomorphism $f : X \to X$ of a metric space X is *expansive* if there is $\epsilon > 0$ such that $x = y$ whenever $x, y \in X$ satisfy $d(f^n(x), f^n(y)) \le \epsilon$ for every $n \in \mathbb{Z}$. Equivalently, if $\Gamma_\epsilon(x) = \{x\}$ for all $x \in X$ where

$$\Gamma_\epsilon^f(x) = \{y \in X : d(f^n(x), f^n(y)) \le \epsilon \text{ for all } n \in \mathbb{Z}\}.$$

Several generalizations of this definition have been appearing in the literature. Of particular interest are the following ones: We say that f is *countably expansive* if there is $\epsilon > 0$ such that $\Gamma_\epsilon^f(x)$ is countable for all $x \in X$. Moreover, f is *measure-expansive* if every non-atomic Borel probability measure μ of X is expansive, namely, there is $\delta > 0$ such that $\mu(\Gamma_\delta^f(x)) = 0$ for every $x \in X$. *Non-atomic* means $\mu(\{x\}) = 0$ for every $x \in X$. In their recent note [2] Artigue and Carrasco-Olivera proved the equivalence between these notions: A homeomorphism of a complete separable metric space is countably expansive if and only if it is measure-expansive.

In this paper we will discuss a similar equivalence but for (continuous) flows. In such a context a flow $\phi : X \times \mathbb{R} \to X$ on X called *measure-expansive* if every ϕ-orbit-vanishing measure is expansive [4]. Recall that a Borel probability measure μ of X is an expansive measure of ϕ if there is $\delta > 0$ such that $\mu(B) = 0$ for every measurable subset $B \subset \Gamma_\delta^\phi(x))$ and every $x \in X$ where

$$\Gamma_\delta^\phi(x) = \{y \in X : d(\phi_t(x), \phi_{c(t)}(y)) \le \delta \text{ for some } c \in \mathcal{C} \text{ and all } t \in \mathbb{R}\}$$

and $\mathcal{C}$ stands for the set of continuous maps $c : \mathbb{R} \to \mathbb{R}$ fixing 0. Also μ is *ϕ-orbit-vanishing* if $\mu(\phi_\mathbb{R}(x)) = 0$ for all $x \in X$ where $\phi_\mathbb{R}(x) = \{\phi_t(x) : t \in \mathbb{R}\}$ denotes the orbit of x.

On the other hand, there is no definition of countably expansive flow in the literature yet. The less general definition of N-expansive flow was managed by Cordeiro [5] (see also [1]). Let N, $Hom(\mathbb{R}, 0)$ and $C^0(A, \mathbb{R})$ be a positive integer, the set of homeomorphisms of $\mathbb{R}$ fixing 0 and the set of continuous maps from $A \subseteq X$ to $\mathbb{R}$ respectively. We say that ϕ is *N-expansive* if for every $\epsilon > 0$ there is $\delta > 0$ such that for every compact subset $A \subseteq X$ and every map $\alpha : A \to Hom(\mathbb{R}, 0)$ satisfying $\alpha(\cdot)(t) \in C^0(A, \mathbb{R})$, $\alpha(x_\alpha) = id$ (identity of $\mathbb{R}$) for some $x_\alpha \in A$ and

$$diam(\{\phi_{\alpha(x)(t)}(x) : x \in A\}) \le \delta, \qquad \forall t \in \mathbb{R},$$

there is $B \subseteq A$ with at most N elements such that $A \subseteq \bigcup_{x \in B} \phi_{(-\epsilon,\epsilon)}(x)$.

Of course it is possible to modify this definition to obtain a notion of countably expansive flow (just demand B above to be at most countable). However, we will manage the following one which looks simpler.

Definition 1.1. We say that ϕ is *countably expansive* if there is $\delta > 0$ (called expansivity constant) such that for any $x \in X$ and $c \in \mathcal{C}$ there exists an at most countable subset $B \subseteq X$ such that $\Gamma_{\delta,c}^\phi(x) \subseteq \bigcup_{x \in B} \phi_\mathbb{R}(x)$, where

$$\Gamma_{\delta,c}^\phi(x) = \bigcap_{t \in \mathbb{R}} \phi_{-c(t)}(B[\phi_t(x), \delta]).$$

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