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A large deviations principle for stochastic flows of viscous fluids

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Abstract

We study the well-posedness of a stochastic differential equation on the two dimensional torus $\mathbb{T}^2$, driven by an infinite dimensional Wiener process with drift in the Sobolev space $L^2\left(0, T; H^1\left(\mathbb{T}^2\right)\right)$. The solution corresponds to a stochastic Lagrangian flow in the sense of DiPerna Lions. By taking into account that the motion of a viscous incompressible fluid on the torus can be described through a suitable stochastic differential equation of the previous type, we study the inviscid limit. By establishing a large deviations principle, we show that, as the viscosity goes to zero, the Lagrangian stochastic Navier–Stokes flow approaches the Euler deterministic Lagrangian flow with an exponential rate function.

Keywords: Navier–Stokes equations; Euler equations; Stochastic differential equations; Stochastic flows; Lagrangian flows; Large deviations principle

1. Introduction

In the literature there are two major approaches to the study of fluid dynamics: the Eulerian approach, which consists in determining and studying the properties of certain physical quantities such as velocity, pressure, etc., at a certain fixed point x in space and time t; and the so-called Lagrangian approximation, where the fluid is seen as a collection of particles that leave the initial

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position and as time progresses describe trajectories in the plane or space. In this perspective the fluid motion can be described by flow mappings more or less smooth (homeomorphisms, diffeomorphisms, etc.) on the region occupied by the fluid.

According to previous works [6], [7], [8], [9], [18] in certain incompressible viscous fluids, despite the deterministic nature of some physical Eulerian quantities, the evolution of particles is inherently stochastic. Thus, in its Lagrangian formulation, stochastic flows should be considered.

One of the classical problems in fluid mechanics is related with the turbulence, where the knowledge of the behaviour of the fluid under the inviscid limit transition is a key step to understand the phenomenon.

In the Eulerian context, the asymptotic study of solutions of the Navier–Stokes equations when the viscosity converges to zero is a difficult issue, still unsolved in three dimensions and in two dimensions in the case of Dirichlet boundary conditions. On a bi-dimensional domain with periodic or slip boundary conditions, it is known that the solutions of the Navier–Stokes equations converges to the solutions of the Euler equations, when the viscosity tends to zero (see [2], [3], [4] and references therein).

In this paper we address the inviscid limit problem for the incompressible viscous fluids from the stochastic Lagrangian point of view, more precisely, we consider the velocity field u^ϵ given as solution of the Navier–Stokes equations in the 2-dimensional torus $\mathbb{T}^2$

$$\begin{cases} \frac{\partial u^\epsilon}{\partial t} + (u^\epsilon.\nabla)u^\epsilon = \epsilon\Delta u^\epsilon + \nabla p \\ \nabla.u^\epsilon = 0 \\ u^\epsilon(x,0) = u_0(x), \;\; u_0 \in H^1(\mathbb{T}^2) \end{cases}$$

and according to [6], we define the stochastic flows as solution of the stochastic differential equation

$$dX_t^\epsilon(x) = u^\epsilon\left(X_t^\epsilon(x),t\right)dt + \sqrt{\epsilon}\sigma(X_t^\epsilon(x))dW_t, \;\; X_0^\epsilon = x, \;\; x \in \mathbb{T}^2.$$

Then our main goal is to establish a Schilder's type theorem, in the sense of Freidlin and Wentzell, for the asymptotic behaviour of the Navier–Stokes flows X_t^ϵ, when the viscosity $\epsilon \to 0$. This asymptotic result shows the exponential concentration of the viscous fluid of Navier–Stokes around the non-viscous fluid of Euler as the viscosity vanishes. Since this probabilist study allows to analyze the probability of rare events, we hope that our approach can give some new insight towards the understanding of turbulent fluids.

Our strategy to prove this asymptotic result is based on the equivalence between the Large deviations principle and the Laplace–Varadhan principle [24] conjugated with the weak convergence approach developed by P. Dupuis and R.S. Ellis [13].

Let us mention that in the study of Large deviations, the well-posedness of the involving dynamic equations is an essential requirement. Here, we deal with stochastic differential equations in which the drift, being a weak solution to the Navier–Stokes equations, does not satisfy the Lipschitz condition. On the other hand, the stochastic noise W_t is an infinite dimensional cylindrical Wiener process and the diffusion is a non-constant Hilbert–Schmidt operator. In this irregular context, the existence and uniqueness of solutions do not come from the classical methods. Then we will follow the theory of DiPerna Lions [12]. Such theory is based on the analysis of the corresponding transport equation and has been applied to stochastic transport equations with Hölder continuous drift and constant diffusion matrix in [17]. However, for non-constant diffusion operators and irregular drift, the study of the corresponding stochastic transport equations is not

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