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Small noise and long time phase diffusion in stochastic limit cycle oscillators

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Abstract

We study the effect of additive Brownian noise on an ODE system that has a stable hyperbolic limit cycle, for initial data that are attracted to the limit cycle. The analysis is performed in the limit of small noise – that is, we modulate the noise by a factor $\varepsilon \searrow 0$ – and on a long time horizon. We prove explicit estimates on the proximity of the noisy trajectory and the limit cycle up to times $\exp\left(c\varepsilon^{-2}\right)$, $c > 0$, and we show both that on the time scale ε^{-2} the *dephasing* (i.e., the difference between noiseless and noisy system measured in a natural coordinate system that involves a phase) is close to a Brownian motion with constant drift, and that on longer time scales the dephasing dynamics is dominated by the drift. The natural choice of coordinates, that reduces the dynamics in a neighborhood of the cycle to a rotation, plays a central role and makes the connection with the applied science literature in which noisy limit cycle dynamics are often reduced to a diffusion model for the phase of the limit cycle.

MSC: 60H10; 34F05; 60F17; 82C31; 92B25

Keywords: Stochastic differential equations; Stable hyperbolic limit cycles; Isochrons; Small noise limit; Long time dynamics

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1. Introduction

1.1. Noise induced dephasing phenomena

Periods, cycles, rhythms are omnipresent and play a fundamental role. In fact, dynamical models proposed in a variety of fields display asymptotically stable periodic behavior, i.e. trajectories are attracted by a periodic trajectory. Important examples come from ordinary differential equations (ODE) with stable limit cycles, like the ODE systems for pray-predator dynamics [28, Ch. 3], and many more examples from life science described in [28], such as gene networks and neural systems. Examples appear also in physics, chemistry and other sciences [10,15,26,32,36]. It is often the case that the ODE model is the result of averaging and/or neglecting plenty of details of the original system that would be more faithfully modeled by keeping a huge number of degrees of freedom. Introducing noise is therefore a way to go a step closer to reality. It is then natural to think of the noise as *small*, for example when one is considering the dynamics of *macroscopic* quantities, i.e. averages of quantities of interest over a whole population. This raises the question, what is the effect of noise on this type of limit cycles?

This is of course not a novel question and it has been often tackled aiming at *reducing the system to a phase*. It is known in fact that in absence of noise and in the proximity of the limit cycle such ODE systems can be reduced to the dynamics of a phase. Furthermore, the system can be mapped to a constant speed rotation on the unit circle [21]. It is therefore natural to seek for phase reductions also in the stochastic setting and a phase reduction for stochastic systems is proposed for example in [26] and has been employed in a number of contexts, see for example the references in [37]. In [37] it has been pointed out that the stochastic phase reduction model that has been used is not accurate and that the noise, even when it is white, induces a *frequency shift*. In [37] a formal small noise development of the solution is given. Clearly, since the noise is weak the leading order behavior – what we may call the *macroscopic behavior* – is just the noiseless behavior. The purpose of [37] and of much of the literature (similar analyses in fact are developed for example in [10, Ch. 6] and [34, § 10.2], with plenty of references) has been on catching both the stochastic and deterministic leading order corrections. Our purpose is to put these works on rigorous and more quantitative grounds, changing somewhat the perspective. The question is rather: on which time scale the difference between the phase dynamics in the noisy and noiseless systems becomes macroscopic and, on this time scale, what is the dynamics? The answer, and central result of our contribution, is that the scale is ε^{-2} if the additive stochastic term is proportional to $\varepsilon > 0$, and that the dephasing dynamics in the limit $\varepsilon \searrow 0$ is a Brownian motion with a constant drift, which corresponds to the frequency shift described in [37].

It is at this point important to stress the following mathematical contributions:

- In [2] the issue of noisy limit cycles is taken up, but the results (see in particular [2, Th. 5.2.3]) are limited to $o\left(\varepsilon^{-2}\right)$ times: the leading order correction to the deterministic cycle on this scale of time is captured, but it is simply given by a Brownian motion and the noise induced drift is absent. This is not in contradiction with our result: a Brownian motion with a drift at infinitesimal times is, to leading order, just a Brownian motion.
- A limit cycle can be seen as an invariant manifold and detailed mathematical work has been developed for systems with a stable hyperbolic manifolds of stationary solutions (see [25, 19] to cite only the finite dimensional cases). These works do analyze the time scale ε^{-2} capturing a noise induced drift that is the analog of what we find. In [31] the limit diffusion process (notably the noise induced drift) is worked out more explicitly than in [25,19] and in

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