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On reconstruction from discrete local moving averages on locally compact abelian groups

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A R T I C L E I N F O

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A B S T R A C T

Let G be a locally compact abelian group and μ be a compactly supported discrete measure on G. We analyse the range of the operator $C_\mu : C(G) \longrightarrow C(G)$ defined by $C_\mu(f)(x) = (f \star \mu)(x) = \int_G f(x-y)d\mu(y)$. It is shown that this operator is onto when G is a compactly generated locally compact abelian group and μ satisfies certain compatibility conditions. Furthermore, if G is a compactly generated torsion free locally compact abelian group then the convolution operator is always onto for every non zero compactly supported discrete measure μ. For a $g \in C(G)$, we construct a function $f \in C(G)$ such that $f \star \mu = g$.

1. Introduction

Certain physical phenomenons which occur in electrical circuit analysis, dynamical systems, economics, statistics, biology and computing etc. are modelled using convolution type equations. For instance, in finding the impulse response of a continuous time systems, the impulse puts signal energy into the system and then goes away. After the impulse energy is injected into the system it responds with a continuous time signal determined by its dynamic character. This model can be written in the form

$$C_\mu(f) = g, \tag{1.1}$$

where g is the observed signal which is the local moving average $C_\mu(f) = \int_G f(x-y)dy$ of the unknown signal f. For a given compactly supported regular Borel measure μ, one easily verifies that $C_\mu(f)$ is a continuous function on G and $C_\mu : C(G) \longrightarrow C(G)$ is a bounded linear operator. The study of such convolution operators was initiated by J. Delsarte [2]. The functions in the kernel of such operators are called mean-periodic functions. The characterization of the kernel when $G = \mathbb{R}$ was given by Laurent Schwartz [7] and on locally compact groups by many authors [1,6,8,9].

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We are interested in knowing the range of the operator C_μ and furthermore, given a function g in the range of C_μ, how to construct an f such that $C_\mu(f) = g$.

Certain special cases of equation (1.1) on the real line and on the plane are analysed in [3,4]. In the present work, we extend these for compactly generated locally compact Abelian groups. It is shown in this paper that the local discrete average operator C_μ maps the space of continuous functions $C(G)$ onto $C(G)$ if the projection of support of μ onto $\mathbb{R}^n \times \mathbb{Z}^m$ via isomorphism is a distinct set. As a special case we obtain that the operator C_μ is onto on $C(G)$ when G is isomorphic to any of the groups: $\mathbb{R}^n$, $\mathbb{Z}^m$ and $\mathbb{R}^n \times \mathbb{Z}^m$. However C_μ is not necessarily onto on $C(G)$ when G has a compact subgroup and the projection of support of μ onto $\mathbb{R}^n \times \mathbb{Z}^m$ via isomorphism is not a distinct set.

2. Notations and preliminaries

We say a subset $C \subseteq \mathbb{R}^n \times \mathbb{Z}^m$ is a pointed convex pseudo cone if $C = \tilde{C} \bigcap (\mathbb{R}^n \times \mathbb{Z}^m)$ for some pointed convex cone $\tilde{C} \subseteq \mathbb{R}^{n+m}$. If G is a locally compact finitely generated torsion free abelian group, then by structure theorem [5], $G \simeq \mathbb{R}^n \times \mathbb{Z}^m$. Let $\varphi : G \longrightarrow \mathbb{R}^n \times \mathbb{Z}^m$ be an onto isomorphism. A set $C_\varphi \subseteq G$ is called a pointed convex pseudo cone with respect to φ if $C_\varphi = \varphi^{-1}(\tilde{C})$ for some pointed convex pseudo cone $\tilde{C}$ in $\mathbb{R}^n \times \mathbb{Z}^m$.

A subset H of $\mathbb{R}^n \times \mathbb{Z}^m$ is called a pseudo hyperplane in $\mathbb{R}^n \times \mathbb{Z}^m$ if $H = \tilde{H} \bigcap (\mathbb{R}^n \times \mathbb{Z}^m)$ for some hyperplane $\tilde{H}$ in $\mathbb{R}^{n+m}$. A left half space in $\mathbb{R}^n \times \mathbb{Z}^m$ is a subset of the form $H^- = \tilde{H}^- \bigcap (\mathbb{R}^n \times \mathbb{Z}^m)$ and a right half space in $\mathbb{R}^n \times \mathbb{Z}^m$ is a subset of the form $H^+ = \tilde{H}^+ \bigcap (\mathbb{R}^n \times \mathbb{Z}^m)$, where $\tilde{H}$ is a hyperplane in $\mathbb{R}^{n+m}$.

A pseudo hyperplane and left and right half spaces in G with respect to an isomorphism φ are defined as the subsets $H_\varphi = \varphi^{-1}(\tilde{H}), H_\varphi^- = \varphi^{-1}(\tilde{H^-})$ and $H_\varphi^+ = \varphi^{-1}(\tilde{H^+})$ respectively, where $\tilde{H}$ is a hyperplane in $\mathbb{R}^n \times \mathbb{Z}^m$.

Definition 2.1. A compactly supported Borel measure μ on G is said to be a discrete Borel measure, if μ can be written as $\mu(E) = \sum_{i=1}^{k} c_i \delta_{x_i}(E)$ for every Borel set E, where $x_1, x_2, \ldots, x_n \in G$ are distinct and $c_1, c_2, \ldots, c_k$ are complex constants. The set of all compactly supported discrete Borel measures on G is denoted by $M_{cd}(G)$.

Definition 2.2. The convolution of a function $f \in C(G)$ with a compactly supported regular Borel measure $\mu \in M_c(G)$ is defined as

$$(f \star \mu)(x) = \int_G f(x-y)d\mu(y).$$

For compactly supported discrete measures of the form $\mu = \sum_{i=1}^{k} c_i \delta_{x_i}$, the above convolution becomes

$$(f \star \mu)(x) = \sum_{i=1}^{k} c_i f(x - x_i).$$

Therefore $f \star \mu = g$ if and only if $\sum_{i=1}^{k} c_i f(x - x_i) = g(x)$ for every $x \in G$.

If a locally compact Abelian group G is compactly generated, then by structure theorem [5], $G \cong \mathbb{R}^n \times \mathbb{Z}^m \times F$ for some compact group F and non negative integers m and n. We denote by Π the projection map from $\mathbb{R}^n \times \mathbb{Z}^m \times F$ to $\mathbb{R}^n \times \mathbb{Z}^m$.