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A critical point theorem in bounded convex sets and localization of Nash-type equilibria of nonvariational systems

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ABSTRACT

The localization of a critical point of minimum type of a smooth functional is obtained in a bounded convex conical set defined by a norm and a concave upper semicontinuous functional. A vector version is also given in order to localize componentwise solutions of variational systems. The technique is then used for the localization and multiplicity of Nash-type positive equilibria of nonvariational systems. Applications are given to periodic problems.

1. Introduction

Many equations and systems arising from mathematical modeling require positive solutions as acceptable states of the investigated real processes. Mathematically, finding positive solutions means to work in the positive cone of the space of all possible states. However, a cone is an unbounded set and in many cases nonlinear problems have several positive solutions. Thus it is important to localize solutions in bounded subsets of a cone. There are known methods for the localization of solutions based on topological fixed point theory [6], [8]; Leray–Schauder degree theory [6]; upper and lower solutions, maximum principles and differential inequalities [2–4], [21]; and critical point theory [1], [5], [7], [12], [15–17], [20], [22], [23]. In case of problems having a variational structure, that is, whose solutions are critical points of an 'energy' functional, the variational techniques are of particular interest since they are able not only to prove the existence of solutions but also to give information about the variational properties of the solutions of a physical relevance, for instance, of being a minimizer, a maximizer or a saddle point of the associated energy

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functional. As known from the classical Fermat's theorem, local extrema of a differentiable functional in a bounded region are not necessarily critical points of that functional. However, this happens if the functional has an appropriate behavior on the boundary of the region (see [12], [15], [22] and [23]).

The problem becomes even more interesting in case of a system which has not a variational structure, but each of its component equations has, i.e., there exist real functionals E_1, E_2 such that the system is equivalent to the equations

$$\begin{cases} E_{11}(u,v) = 0 \\ E_{22}(u,v) = 0 \end{cases}$$

where $E_{11}(u,v)$ is the partial derivative of E_1 with respect to u, and $E_{22}(u,v)$ is the partial derivative of E_2 with respect to v. How the solutions (u,v) of this system are connected to the variational properties of the two functionals? One possible situation, which fits to physical principles, is that a solution (u,v) is a Nash-type equilibrium of the pair of functionals (E_1, E_2) (see, e.g., [9], [13] and [24]), that is

$$E_1(u,v) = \min_w E_1(w,v)$$
$$E_2(u,v) = \min_w E_2(u,w).$$

A result in this direction is given in [18] for the case when $\min_w$ is taken, first over an entire Banach space and then, over a ball. Non-smooth analogues of those results, for Szulkin functionals, are presented in [19].

In the present paper the localization of a Nash-type equilibrium (u,v) is obtained in the Cartesian product of two conical sets, more exactly $u \in K_1$, $v \in K_2$ where K_i $(i = 1, 2)$ is a cone of a Hilbert space X_i with norm $\|\cdot\|_i$, and

$$r_1 \le l_1(u), \qquad \|u\|_1 \le R_1,$$
$$r_2 \le l_2(v), \qquad \|v\|_2 \le R_2,$$

for some positive numbers r_i and R_i, $i = 1, 2$. Here $l_i : K_i \to \mathbb{R}_+$ are two given functionals. Compared to our previous papers on the localization of critical points in annular conical sets (see [15–17] and [20]), where l_i were norms, here they are upper semicontinuous concave functionals. In applications, when working in spaces of functions, such a functional $l(u)$ can be $\inf u$. If in addition, due to some embedding result, the norm $\|u\|$ is comparable with $\sup u$ in the sense that $\sup u \le c\|u\|$ for every nonnegative function u and some constant $c > 0$, then the values of any nonnegative function u satisfying $r \le l(u)$ and $\|u\| \le R$ belong to the interval $[r, cR]$, which is very convenient for finding multiple solutions located in disjoint annular conical sets.

The paper is structured as follows: first in Section 2 we establish the localization of a critical point of minimum type in a convex conical set as above and we explain how this result can be used in order to obtain finitely or infinitely many solutions. The result can be seen as a variational analogue of some Krasnoselskii's type compression–expansion theorems from fixed point theory (see, e.g., [8], [10] and [11]). The vector version of this result for gradient type systems is obtained in Section 3. It allows to localize individually the components of a solution. Section 4 is devoted to the existence and localization of Nash-type equilibria for nonvariational systems of two equations. An iterative algorithm is used and its convergence is established assuming a **local** matricial contraction condition. The local character of the contraction condition makes possible a repeat application of the algorithm to a number of disjoint conical sets and thus the obtainment of multiple Nash-type equilibria. The theory developed in Sections 2, 3 and 4 is illustrated in Section 5 on the periodic problem.