J. Math. Anal. Appl. 462 (2018) 554–576

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Derivative formulae for stochastic differential equations driven by Poisson random measures ☆

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ARTICLE INFO

Article history:
Received 7 October 2016
Available online 13 February 2018
Submitted by U. Stadtmueller

Keywords:
Stochastic differential equations
Lévy processes
Poisson functional
Dirichlet form
Derivative formulae
Gradient estimate

ABSTRACT

We establish some new derivative formulae of Bismut–Elworthy–Li's type for stochastic differential equations with respect to Poisson point processes using the lent particle method created by Bouleau and Denis.

1. Introduction

Let $(\Xi, \mathcal{G}, \nu)$ be a measurable space and $f : [0, 1] \times \mathbb{R}^d \times \Xi \to \mathbb{R}^d$, $g : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^n$. Let $\mathcal{B}_b(\mathbb{R}^d)$ be the set of all bounded measurable functions on $\mathbb{R}^d$. Define

$$P_t\varphi(x) := \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in \mathcal{B}_b(\mathbb{R}^d), \quad t \in [0, 1],$$

where $X(t, x)$ solves the following stochastic differential equation (SDE in short) with jumps:

☆ Work supported by the National Natural Science Foundation of China (Grant Nos. 11461028, 11471340 and 11671408), Postdoctoral Science Foundation of China (Grant No. 2015M572128), Natural Science Foundation of Jiangxi Province for Young Scholars (Grant No. 20161BAB211019) and Educational Commission Science Programm of Jiangxi Province (Grant No. GJJ150445).

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https://doi.org/10.1016/j.jmaa.2018.02.026
0022-247X/

$$X(t,x) = x + \int_0^t \int_\Xi f(t, X(t-,x), y)\widetilde{N}(dt,dy) + \int_0^t g(t, X(t-,x))Z(dt), \quad t \in [0,1], \tag{1}$$

where $\widetilde{N}$ is the compensated process of some Poisson point process N on $(\Xi, \mathcal{G})$ with intensity ν and Z a semimartingale independent of N. For more precise meaning see Section 2 below. $\{P_t; t \in [0,1]\}$ is usually called the Markov semigroup associated with $X(t,x)$. In this paper, we aim to establish a derivative formula of Bismut–Elworthy–Li's type:

$$\nabla P_t \varphi(x) = \mathbb{E}[\varphi(X(t,x))\Psi(x)], \quad \varphi \in \mathcal{B}_b(\mathbb{R}^d),$$

where ∇ is the weak or distributional derivatives, and $\Psi(x)$ is a $\mathbb{R}^d$-valued stochastic process.

In the recent years, quite a few papers appeared concerning the derivative formulae for SDEs with jumps, e.g., SDEs driven by general Poisson jumps processes by using stochastic diffeomorphism flows and Girsanov's transformation ([9]), SDEs driven by subordinated Brownian motion by using conditional Malliavin calculus ([13,12]), SDEs driven by α-stable-like noise by using Bismut's approach ([11]). In these works, it was assumed that the Lévy intensity measure ν has finite moments of all orders, or satisfies some growth condition, and hence some interesting cases were ruled out.

Recently, Bouleau and Denis developed systematically a new method called the lent particle method in Malliavin calculus with jumps, which allows to remove some restrictions on the intensity in most previous approaches. The calculus with this approach is more convenient since the gradient operator is local (see [4] and references therein). Making use of this method, we establish in this paper derivative formulae of Bismut–Elworthy–Li's type for SDEs with jumps. We first obtain a derivative formula for general SDEs driven by Poisson random measures. Then in the special case of SDEs driven by Lévy processes we obtain a more explicit formula which enables us to obtain a gradient estimate for the corresponding semigroup. Compared with the previous results in the existing literature, the restrictive conditions on the existence of moments of all orders and on the growth of Lévy measure will be relaxed and we will have a flexible choice for the Dirichlet structure on the bottom space, which would be useful in applications.

The organization of this paper is as follows. We devote Section 2 to recalling the lent particle method. In Section 3, we obtain a derivative formula for general SDEs with jumps. Section 4 and 5 are devoted to deducing the derivative formula for SDEs driven by a Lévy process and obtaining the gradient estimate for the corresponding semigroups.

2. Set-up

Let us first specify the general set-up in which we will work. We follow [4] to which we refer for more details.

2.1. Dirichlet structure on the bottom space

We start from a bottom space $(\Xi, \mathcal{G}, \nu)$, where Ξ is a separable Hausdorff space, $\mathcal{G}$ its Borel σ-algebra and ν a σ-finite and diffuse measure on $(\Xi, \mathcal{G})$. Let $(\mathbf{d}, e)$ be a local symmetric Dirichlet form on $L^2(\nu)$ which admits a carré du champ operator γ. That is to say, γ is the unique positive, symmetric and continuous bilinear form from $\mathbf{d} \times \mathbf{d}$ to $L^1(\nu)$ such that

$$e(f,g) = \frac{1}{2}\int \gamma[f,g]d\nu, \quad \forall f,g \in \mathbf{d}.$$

Moreover, we suppose that there exist $\{k_n, n \in \mathbb{N}\} \subset \mathbf{d}$ and $A_n \uparrow \mathcal{X}$ such that, $k_n 1_{A_n} \uparrow 1$ and $\gamma[k_n]1_{A_n} = 0$. The structure $(\Xi, \mathcal{G}, \nu, \mathbf{d}, \gamma)$ is called the bottom structure.