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The polynomial cluster value problem

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ABSTRACT

The polynomial cluster value problem replaces the role of the continuous linear functionals in the original cluster value problem for the continuous polynomials to describe the corresponding cluster sets and fibers. We prove several polynomial cluster value theorems for uniform algebras $H(B)$ between $A_u(B)$ and $H^\infty(B)$, where B is the open unit ball of a complex Banach space X. We also obtain new results about the original cluster value problem, especially for $A_\infty(B)$. Examples of spaces X considered here are spaces of continuous functions, ℓ_1 and locally uniformly convex spaces.

1. Introduction

The original cluster value problem concerns the boundary behavior of complex-valued, bounded, holomorphic functions on the open unit ball B of a complex Banach space X. The space of bounded holomorphic functions forms the unital, commutative Banach algebra $H^\infty(B)$. This is, however, a very large algebra, so it is natural to work with its more manageable unital subalgebras $H(B)$ containing the bounded linear functionals X^*. For example, $A(B)$ is the subalgebra of $H^\infty(B)$ generated precisely by $X^*|_B$ and the constant 1.

For a commutative Banach algebra $H(B)$ as before, there is a norm-decreasing homomorphism from $H(B)$ to the continuous functions on its spectrum, called the Gelfand transform. This homomorphism results an isometric embedding since the norm of this Banach algebra equals its spectral radius. As it turns out, the cluster value problem can be generally posed in terms of the spectrum of $H(B)$. This spectrum consists of

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the nonzero multiplicative linear functionals on $H(B)$, denoted $M_{H(B)}$. Unital and commutative Banach algebras $H(B)$ have its spectrum contained in the norm-one linear functionals $S_{H(B)^*}$, and this spectrum is a nonempty compact space in the usual weak-star topology making it a subspace of the unit ball of $H(B)^*$. Obvious members in the spectrum of $H(B)$, also called its characters, are evaluations at points of B. Other characters can be found through its respective kernel, e.g. when it is the maximal ideal containing the functions that nullify over a certain limit [17, §10.1], [5, §7], [7, Proposition 1.5].

Identifying the whole spectrum seems out of reach, but there have been advances discovering its structure. For example, the spectrum of H^∞ of the unit disk $\mathbb{D}$ can be naturally mapped to the closed unit disk by evaluating each character at the identity function $z \mapsto z$. It is known that this mapping π is continuous and surjective, and it is injective on the inverse image of the open unit disk. The remainder of the spectrum is mapped to the unit circle, so for each $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ we call $\pi^{-1}(\alpha)$ the fiber over α of the spectrum. In 1961 I.J. Schark discovered that, for each f in H^∞ of the unit disk $\mathbb{D}$, the range of the Gelfand transform $\hat{f}$ on the fiber over any unitary $\alpha \in \mathbb{C}$ coincides with the cluster set of f at x_0, i.e. the limit values of $f(x)$ as $x \in \mathbb{D}$ tends to x_0 in norm [25].

The last result was surpassed in 1962 by the Corona theorem of Carleson. The Corona of the spectrum of a Banach algebra $H(B)$ consists of those characters that are not in the closure of the ball B. The Corona problem asks whether such Corona is empty and, if so, we say we have a Corona theorem. Carleson proved that the Corona of $H^\infty(\mathbb{D})$ is empty [10]. Fundamental results about the Corona problem appear in [12].

In the next decade, respective cluster set identifications were established despite the corresponding Corona staying unknown: by Gamelin in 1973 for the algebra H^∞ of the polydisk in $\mathbb{C}^n$ [15], and by McDonald in 1979 for H^∞ of the Euclidean unit ball in $\mathbb{C}^n$ or any other strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary [20]. The latter result partially relied on the existence of peaking functions at points of strong pseudoconvexity [24].

Passing from $\mathbb{C}^n$ to an arbitrary ambient space X, the cluster set of $f \in H(B)$ at x^{**} in the closed ball of the bidual $\bar{B}^{**}$ consists of the accumulation points of values $f(x)$ as $x \in B$ tends to x^{**} in the weak-star topology. Note that the weak-star topology in X^{**} is the initial topology with respect to X^*, $w(X^{**}, X^*)$, and given that $H(B)$ contains the algebra generated by X^* and 1, we consider the mapping π restricting each character to X^* to become a character of $A(B)$, and again partition the spectrum of $H(B)$ in fibers via inverse images of π, ending up with each fiber associated to a point of $\bar{B}^{**}$. When the Corona of $H(B)$ is empty, once again the boundary behavior of each $f \in H(B)$ is determined: the cluster set of $f \in H(B)$ at every $x^{**} \in \bar{B}^{**}$ coincides with the Gelfand transform $\hat{f}$ evaluated in the fiber over x^{**} [2].

A cluster value theorem for the Banach algebra $H(B)$ at the point $x^{**} \in \bar{B}^{**}$ asserts that the previously described cluster set identification holds (even if we do not know whether the Corona is empty). If this cluster set characterization is moreover true for all $x^{**} \in \bar{B}^{**}$ then there is a cluster value theorem for $H(B)$. The cluster value problem was first set up in this generality in [2]. Several broad ideas about the cluster value problem were first established in [2], [18] and [3], and the first infinite-dimensional cluster value theorems were proved in [2] and [19] focusing on Banach algebras on the ball of some classical Banach spaces. Examples include the uniformly continuous and holomorphic functions $A_u(B)$ on the ball B of any Hilbert space, and H^∞ for the ball of c_0 or any space of continuous functions on a scattered compact Hausdorff space. The state of the art on the cluster value problem was surveyed in [9].

It is worth mentioning that for finite-dimensional spaces, c_0 and the spaces of continuous functions on scattered compact Hausdorff spaces K the algebra $A_u(B)$ coincides with the ball algebra $A(B)$. In contrast, as soon as K is not scattered, the algebras $A(B)$ and $A_u(B)$ are no longer equal [19]. Since $A_u(B)$ is the subalgebra of $H^\infty(B)$ generated by the continuous polynomials on X, $P(X)$ [5, p. 56], while $A(B)$ is generated by $X^*|_B$ and 1, it was first considered in [19, p. 1565] to specialize in the *cluster value problem for $H^\infty(B)$ over $A_u(B)$* in which the weak-star topology is replaced by the polynomial-star topology (to be described in Section 2) and the mapping π is replaced by the restriction $\pi^{\mathcal{P}}$ of characters to $A_u(B)$. In this paper we analyze the polynomial cluster value problem just described but for any algebra $H(B)$ between