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On Picard's theorem

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A R T I C L E I N F O

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A B S T R A C T

We give a connection/equivalence between Picard's theorem and characterization of entire solutions of a differential equation.

1. Picard's theorem asserts that an entire function, i.e., a complex-valued function differentiable in the complex plane **C**, omitting two complex numbers must be constant. It also implies, by a linear transform, the meromorphic version of the theorem that a meromorphic function in **C** omitting three distinct values must be constant. Picard's theorem is among the most striking results in complex analysis and plays a decisive role in the development of the theory of entire and meromorphic functions and other applications. It is a significant strengthening of Liouville's Theorem which states that a bounded entire function must be constant. While Liouville's Theorem can be treated as a consequence of Cauchy's formula/theorem, Picard's theorem is generally not encountered until advanced complex analysis involving rather heavy machinery. Different proofs of Picard's theorem are known (see [1–7], [12], etc.). We refer to [10] for an exposition (the history, methods and references) of the theorem.

In this short article, we give a connection/equivalence between Picard's theorem and characterization of entire solutions of a differential equation, which does not seem to have been observed before and may lead further results on Picard type theorems and complex (ordinary and partial) differential equations (cf. §2).

Theorem 1. *Let $a(z)$ be an entire function and let $P(z)$ be a meromorphic function in* **C** *with at least two distinct zeros. Then an entire solution of the differential equation*

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$$f' + a(z)P(f) = 0 \tag{1}$$

must be constant.

We will see that this theorem immediately gives Picard's theorem; as a matter of fact, Theorem 1 and Picard's theorem are equivalent. Before that, we first give counterexamples to show that Theorem 1 is best possible in various aspects in the following

Remark. (i) Theorem 1 does not hold if the function P is assumed to have at most one zero (counting or ignoring multiplicities). For example, let $P(z) = z^n$ ($n \geq 0$ an integer), and $a(z) = -e^{-(n-1)z}$. Then $f = e^z$ is a nonconstant entire solution of the equation (1).

(ii) The function $a(z)$ in Theorem 1 cannot be improved to a meromorphic function. Let $a(z) = -\frac{1}{e^z-1}$ and $p(z) = z(z-1)$. Then $f = e^z$ is a nonconstant entire solution of the equation (1).

(iii) "Entire solution" in Theorem 1 cannot be improved to "meromorphic solution". For example, $f = \frac{1}{e^z-1}$ is a nonconstant meromorphic solution of the equation (1) with $a(z) = 1$ and $p(z) = z(z+1)$.

We now show that Theorem 1 and Picard's Theorem are equivalent in the sense that one implies the other.

Theorem 1 $\Longrightarrow$ Picard's Theorem. Assume that an entire function f omits two distinct complex numbers c, d. Then $a(z) := \frac{f'}{(f-c)(f-d)}$ is entire. Clearly, $f' - a(z)(f-c)(f-d) = 0$. Thus, f must be constant by Theorem 1. □

Picard's Theorem $\Longrightarrow$ Theorem 1. Since the meromorphic function P has at least two distinct zeros, say c, d, we can write $P(z) = (z-c)^m(z-d)^n g(z)$, where m, n are two positive integers and g is a meromorphic function in $\mathbf{C}$ and holomorphic at c and d. We can then write (1) as

$$f'(z) = -a(z)(f(z)-c)^m(f(z)-d)^n g(f(z)). \tag{2}$$

We assert that an entire solution f cannot assume c and d; otherwise the right hand side of (2) would have a zero (coming from a zero of $f-c$ or $f-d$) with multiplicity strictly greater than that of the same zero of the left hand side (due to the derivative, which decreases the multiplicity), which is absurd. Thus, the entire function f omits c and d. By Picard's theorem, f is constant. □

2. Characterizing complex analytic solutions of differential equations is a topic of a long history. In Theorem 1, we are not intended to give the most general differential equations, but rather to expose the connection/equivalence between Picard's theorem and characterization of entire solutions of the differential equation. Theorem 1, in the form about entire solutions of a differential equation, leads two natural questions: Can Theorem 1 be proved independent of Picard's theorem (and thus also provide another proof of Picard's theorem)? Can Theorem 1 be generalized for more general ordinary and even partial differential equations? Here we include such a proof, inspired by the previous work [7], using "pre-Nevanlinna theory" (cf. below), which also shows how Theorem 1 can be pushed over to the following result for partial differential equations

$$\sum_{|\alpha|=1}^{m} a_\alpha \frac{\partial^{|\alpha|} f(z)}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_n} z_n} + a(z)P(f(z)) = 0 \tag{3}$$

where $z = (z_1, z_2, \cdots, z_n)$ in $\mathbf{C}^n$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index with $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and a_α's are constant.