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Short Communication

Simplifying differential equations concerning degenerate Bernoulli and Euler numbers

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Abstract

In the paper, the authors significantly and meaningfully simplify two families of nonlinear ordinary differential equations in terms of the Stirling numbers of the first and second kinds.

Keywords: Nonlinear ordinary differential equation; Stirling number of the first kind; Stirling number of the second kind; Bernoulli number; Euler number

1. Motivations and main results

In this section, we state three motivations and our main results of current paper.

1.1. First motivation

In [1, Theorems 1 and 2], the authors spent five pages to elementarily, recurrently, and inductively prove that the ordinary differential equations

$$F_{\mp}^{(N)} = \left(-\frac{1}{\lambda}\ln(1+\lambda)\right)^{N} \sum_{i=1}^{N+1} a_{i-1}^{\mp}(N) F_{\mp}^{i}, \quad N = 0, 1, 2, \dots \tag{1}$$

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have a solution

$$F_{\mp} = F_{\mp}(t) = \frac{1}{(1+\lambda)^{t/\lambda} \mp 1},$$

where $a_0^{\mp}(N) = 1$ and

$$a_j^{\mp}(N) = (-1)^{(j\mp j)/2} j! \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}} \cdots 2^{i_1} \tag{2}$$

for $1 \le j \le N$. With the aid of the quantities $a_j^{\mp}(N)$ in (2), the authors further obtained in [1, Theorems 3 and 6, Corollaries 4, 5, and 7] several identities and explicit expressions of the modified degenerate Euler numbers $\tilde{\mathcal{E}}_n(\lambda)$, the higher order modified degenerate Euler numbers $\tilde{\mathcal{E}}_n(\lambda)$, the Euler numbers E_n, the higher order Euler numbers E_n, the modified degenerate Bernoulli numbers $\tilde{\beta}_n(\lambda)$, the higher order modified degenerate Bernoulli numbers $\tilde{\beta}_n^{(r)}(\lambda)$, the Bernoulli numbers B_n, and the higher order Bernoulli numbers $B_n^{(r)}$, which can be generated respectively by

$$\frac{2}{(1+\lambda)^{t/\lambda}+1} = \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_n(\lambda) \frac{t^n}{n!}, \qquad \left[\frac{2}{(1+\lambda)^{t/\lambda}+1}\right]^r = \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_n^{(r)}(\lambda) \frac{t^n}{n!},$$

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \qquad \left(\frac{2}{e^t+1}\right)^r = \sum_{n=0}^{\infty} E_n^{(r)} \frac{t^n}{n!},$$

$$\frac{t}{(1+\lambda)^{t/\lambda}-1} = \sum_{n=0}^{\infty} \tilde{\beta}_n(\lambda) \frac{t^n}{n!}, \qquad \left[\frac{t}{(1+\lambda)^{t/\lambda}-1}\right]^r = \sum_{n=0}^{\infty} \tilde{\beta}_n^{(r)}(\lambda) \frac{t^n}{n!},$$

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad \left(\frac{t}{e^t-1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \frac{t^n}{n!},$$

where $r \in \mathbb{N}$. This means that the quantities $a_j^{\mp}(N)$ in (2) play an important role in the paper [1].

It is clear that the quantities $a_j^{\mp}(N)$ in (2) are formulated by j multiple sums. To the best of our knowledge, one cannot understand and compute easily such a j multiple sum. Then we guess that, making use of some different methods from the one employed in the paper [1], the quantities $a_j^{\mp}(N)$ in (2) should be reformulated simply, meaningfully, and significantly in terms of some mathematical quantities.

1.2. Second motivation

In the paper [2], Theorem 1 reads that the function

$$F_q(t) = \frac{1}{qe^t+1}$$

satisfies

$$(N-1)! F_q^N = \sum_{k=1}^{N} a_k(N) F_q^{(k-1)}$$

for $N \ge 1$ and $q \in \mathbb{R}$, where

$$a_k(N) = (-1)^{N+k} s(N,k) \tag{3}$$

or

$$a_k(N) = \frac{N!}{k!} \sum_{\substack{\ell_1,\dots,\ell_k \ge 1 \\ \ell_1+\cdots+\ell_k = N}} \frac{1}{\ell_1+\cdots+\ell_k}, \tag{4}$$

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