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Cyclically symmetric lozenge tilings of a hexagon with four holes

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The work of Mills, Robbins, and Rumsey on cyclically symmetric plane partitions yields a simple product formula for the number of lozenge tilings of a regular hexagon, which are invariant under rotation by 120°. In this paper we generalize this result by enumerating the cyclically symmetric lozenge tilings of a hexagon in which four triangles have been removed in the center.

1. Introduction

A *plane partition* is a rectangular array of non-negative integers $\pi = (\pi_{i,j})$ with weakly decreasing rows and columns. Here $\pi_{i,j}$'s are the *parts* (or the *entries*) of the

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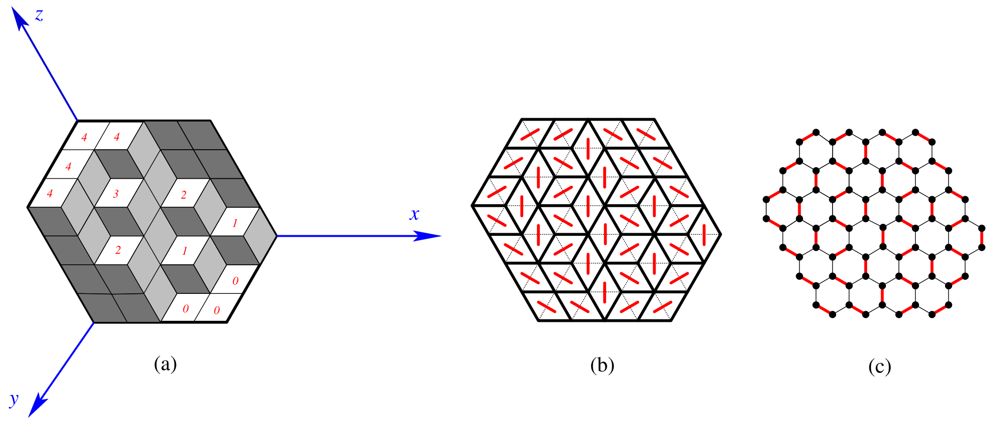

Fig. 1.1. The bijection between plane partitions, lozenge tilings, and perfect matchings. (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

plane partition, and the sum of all the parts $|\pi| = \sum_{i,j} \pi_{i,j}$ is called the *norm* (or the *volume*) of the plane partition. For example, the plane partition

$$\pi = \begin{matrix} 4 & 4 & 2 & 1 \\ 4 & 3 & 1 & 0 \\ 4 & 2 & 0 & 0 \end{matrix}$$

has 3 rows, 4 columns, and the norm 25.

A plane partition with a rows, b columns, and parts at most c is usually identified with its 3-D interpretation, a monotonic stack (or pile) of unit cubes contained in an $a \times b \times c$ box. In particular, the monotonic stack corresponding to the above plane partition π has the heights of the columns of unit cubes weakly decreasing along $\overrightarrow{\mathbf{Ox}}$ and $\overrightarrow{\mathbf{Oy}}$ directions as shown in Fig. 1.1(a). In view of this, we say that the latter plane partition 'fits in an $a \times b \times c$ box.' Each of the monotonic stacks in turn can be viewed as a lozenge tiling of the semi-regular hexagon $\mathcal{H}_{a,b,c}$ with side-lengths a, b, c, a, b, c (in clockwise order, starting from the northwest side) and all angles 120° on the triangular lattice. A *lozenge* is a union of any two unit equilateral triangles sharing an edge, and a *lozenge tiling* of a region is a covering of the region by lozenges so that there are no gaps or overlaps. (See Fig. 1.1(b). We ignore the red intervals at the center of each lozenge at this moment.)

MacMahon [21] proved that the total number of plane partitions that fit in an $a \times b \times c$ box, also the number of lozenge tilings of the hexagon $\mathcal{H}_{a,b,c}$, is equal to

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}. \tag{1.1}$$

It has been shown that various symmetry classes of lozenge tilings of the semi-regular hexagon also yield simple product formulas (see e.g. the classical paper of Stanley [27],