Applied Mathematics and Computation 339 (2018) 451–458

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Some identities of the generalized Fibonacci and Lucas sequences

Jizhen Yang, Zhizheng Zhang*

Department of Mathematics, Luoyang Normal University, Luoyang 471934, PR China

ARTICLE INFO

Keywords:
Generalized Fibonacci sequence
Generalized Lucas sequence
Identity
Congruence

ABSTRACT

The purpose of this paper is to study generalized Fibonacci and Lucas sequences. We first introduce generalized Lucas sequences. Section 2 contains a list of elementary relationships about generalized Fibonacci and Lucas sequences. In Section 3, we give a generalization of the Binet's formulas of generalized Fibonacci, Lucas sequences and its applications. Section 4 is devote to derive many identities and congruence relations for generalized Fibonacci, Lucas sequences by using operator method.

1. Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field of science and art [5]. Fibonacci numbers are frequently extended. For recent generalizations, see, for example, [4,6,8–10,12]. Specially, in [2,11], Edson and Yayenie introduced and studied a new generalized Fibonacci sequence that depends on two real parameters used in a non-linear (piecewise linear) recurrence as defined below.

Definition 1.1. For any two nonzero real numbers a and b, the generalized Fibonacci sequence $\{q_n\}_{n=0}^{\infty}$ is defined recursively by

$$q_0 = 0, \quad q_1 = 1, \quad q_n = \begin{cases} aq_{n-1} + q_{n-2}, & if\ n\ is\ even, \\ bq_{n-1} + q_{n-2}, & if\ n\ is\ odd, \end{cases} \quad (n \geq 2). \tag{1}$$

This generalized Fibonacci sequences have word combinatorial interpretation and they are also closely related to continued fraction expansion of quadratic irrationals (see [2]). Here we introduce the generalized Lucas sequences defined by

Definition 1.2. For any two nonzero real numbers a and b, the generalized Fibonacci sequence $\{t_n\}_{n=0}^{\infty}$ is defined recursively by

$$t_0 = 2a, \quad t_1 = ab, \quad t_n = \begin{cases} at_{n-1} + t_{n-2}, & if\ n\ is\ even, \\ bt_{n-1} + t_{n-2}, & if\ n\ is\ odd, \end{cases} \quad (n \geq 2). \tag{2}$$

From Definitions 1.1 and 1.2, it is easy to see that when $a = b = 1$, we have the classical Fibonacci, Lucas sequences and when $a = b = 2$, we obtain the Pell, Pell–Lucas numbers. If we set $a = b = k$, for some positive integer k, we get the

* Corresponding author.
E-mail addresses: yangjizhen116@163.com (J. Yang), zhzhzhang@lynu.edu.cn (Z. Zhang).

https://doi.org/10.1016/j.amc.2018.07.054
0096-3003/

k-Fibonacci, k-Lucas numbers. If $a=1$ and $b=2$, then members of the sequence q_n are denominators of continued fraction converging to $\sqrt{3}$ (see A002530) in [7].

In [2,11], some properties of the generalized Fibonacci sequences q_n were studied. In this paper, we first introduce generalized Lucas sequences. Section 2 contains a list of elementary relationships about generalized Fibonacci and Lucas sequences. In Section 3, we give a generalization of the Binet's formulas of generalized Fibonacci, Lucas sequences and its applications. Section 4 is devoted to many identities and congruence relations for generalized Fibonacci, Lucas sequences by using operator method.

2. Some preliminary results

In [2,11], Edson and Yayenie gave the following properties of the generalized Fibonacci sequences q_n.

Theorem 2.1. *Let* $\xi(n)=n-2\lfloor\frac{n}{2}\rfloor=\begin{cases}0, & n \text{ even}\\ 1, & n \text{ odd}\end{cases}$. *Then*

(1). *([2, Theorem 5]), Binet's formula:* $q_n=\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor n/2\rfloor}}\right)\frac{\alpha^n-\beta^n}{\alpha-\beta}$, *where* $\alpha=\frac{ab+\sqrt{a^2b^2+4ab}}{2}$ *and* $\beta=\frac{ab-\sqrt{a^2b^2+4ab}}{2}$

(2). *([2, Theorem 4]), Generating function:* $\sum_{n=0}^{\infty}q_nx^n=\frac{a(1+ax-x^2)}{1-(ab+2)x^2+x^4}$;

(3). *([11, p5604]):* $q_{m+2}=(ab+2)q_m-q_{m-2}$;

(4). *([11, Theorem 1]):* $q_{n+6}=(ab+3)a^{1-\xi(n)}b^{\xi(n)}q_{n+3}+q_n$;

(5). *([11, Theorem 3]):* $a^{\xi(mn+n-m)-1}b^{1-\xi(mn+n-m)}q_mq_n+a^{-\xi(mn)}b^{\xi(mn)}q_{m-1}q_{n-1}=q_{m+n-1}$;

(6). *([2, Theorem 10]):* $\sum_{k=0}^{n}\binom{n}{k}a^{\xi(k)}(ab)^{\lfloor\frac{k}{2}\rfloor}q_k=q_{2n}$; $\sum_{k=0}^{n}\binom{n}{k}a^{\xi(k+1)}(ab)^{\lfloor\frac{k+1}{2}\rfloor}q_{k+1}=aq_{2n+1}$;

(7). *([11, Theorem 5]):* $q_n=\frac{a^{\xi(n+1)}}{2^{n-1}}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2i+1}(ab)^{\lfloor\frac{n-1}{2}\rfloor-i}(ab+4)^i$.

Corresponding to Theorem 2.1, according to the method of [2,11], we can get the following results for generalized Lucas sequences t_n.

Theorem 2.2. *Let* $\xi(n)=n-2\lfloor\frac{n}{2}\rfloor=\begin{cases}0, & n \text{ even}\\ 1, & n \text{ odd}\end{cases}$. *Then*

(1). *Binet's formula:* $t_n=\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor n/2\rfloor}}\right)(\alpha^n+\beta^n)$, *where* $\alpha=\frac{ab+\sqrt{a^2b^2+4ab}}{2}$ *and* $\beta=\frac{ab-\sqrt{a^2b^2+4ab}}{2}$

(2). *Generating function:* $\sum_{n=0}^{\infty}t_nx^n=\frac{a(2+bx-(ab+2)x^2+bx^3)}{1-(ab+2)x^2+x^4}$;

(3). $t_{m+2}=(ab+2)t_m-t_{m-2}$;

(4). $t_{n+6}=(ab+3)a^{1-\xi(n)}b^{\xi(n)}t_{n+3}+t_n$;

(5). $a^{\xi(mn+n-m)-1}b^{1-\xi(mn+n-m)}t_mq_n+a^{-\xi(mn)}b^{\xi(mn)}t_{m-1}q_{n-1}=t_{m+n-1}$;

(6). $\sum_{k=0}^{n}\binom{n}{k}a^{\xi(k)}(ab)^{\lfloor\frac{k}{2}\rfloor}t_k=t_{2n}$, $\sum_{k=0}^{n}\binom{n}{k}a^{\xi(k+1)}(ab)^{\lfloor\frac{k+1}{2}\rfloor}t_{k+1}=at_{2n+1}$;

(7). $t_n=\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}2^{n-1}}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2i}(ab)^{m-i}(ab+4)^i$.

Applying recurrence relations of q_n and t_n, we add the following identities.

Theorem 2.3. *We have*

$$q_nt_n=\frac{a^{1-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor-n}}q_{2n},$$

$$\sum_{i=1}^{n}a^{1-\xi(i+1)}b^{\xi(i+1)}q_i=q_{n+1}+q_n-1,$$

$$\sum_{i=1}^{n}a^{1-\xi(i+1)}b^{\xi(i+1)}t_i=t_{n+1}+t_n-a(b+2),$$

$$\sum_{i=1}^{n}q_{2i-1}=\frac{1}{a}q_{2n},\quad \sum_{i=1}^{n}q_{2i}=\frac{1}{b}(q_{2n+1}-1),$$

$$\sum_{i=1}^{n}t_{2i-1}=\frac{1}{a}t_{2n}-2,\quad \sum_{i=1}^{n}t_{2i}=\frac{1}{b}(t_{2n+1}-ab).$$